Kinematics: Fundamentals

EEL 4930/5934: Autonomous Robots

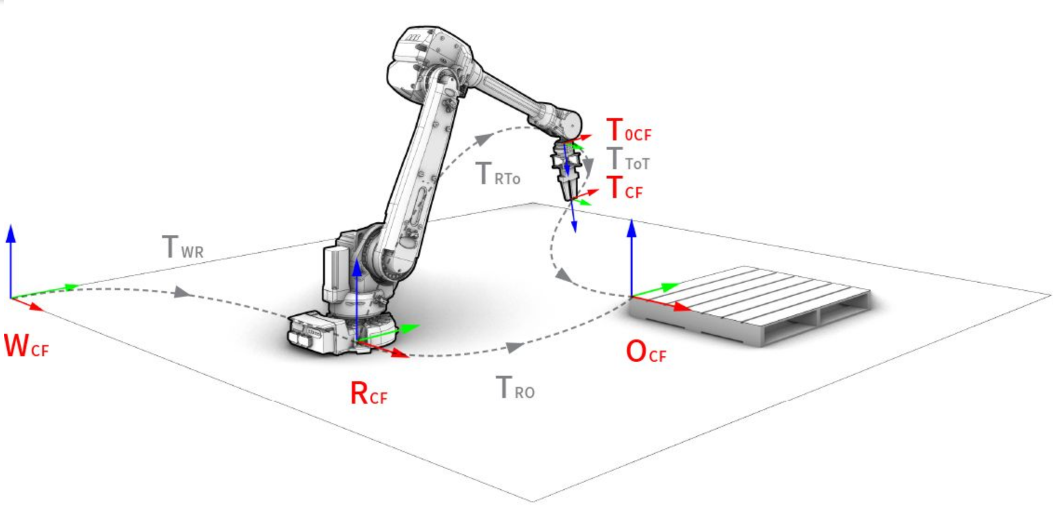
Spring 2023

Md Jahidul Islam

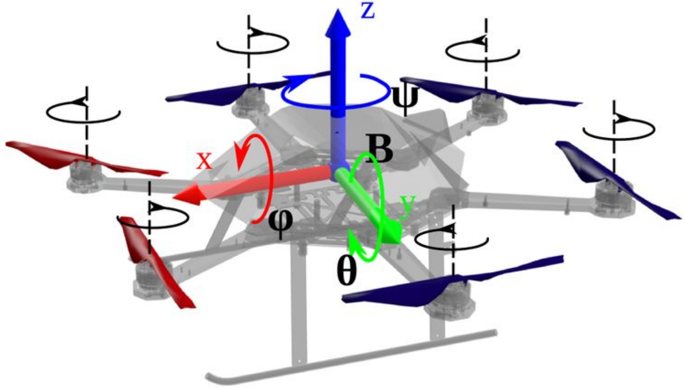
Lecture 3

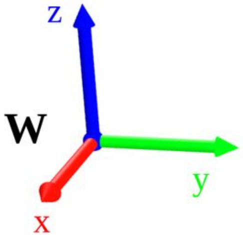
ECE | Florida
Electrical & Computer Engineering
UF | UNIVERSITY of FLORIDA

Coordinate Frame Of Reference



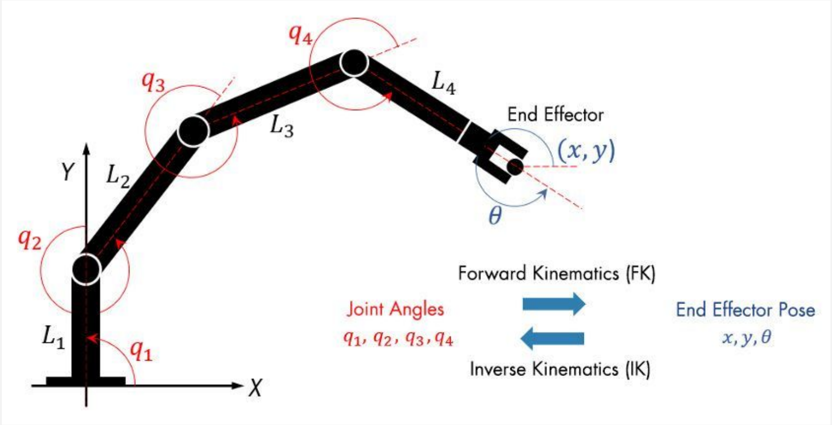
Read: Coordinate frames

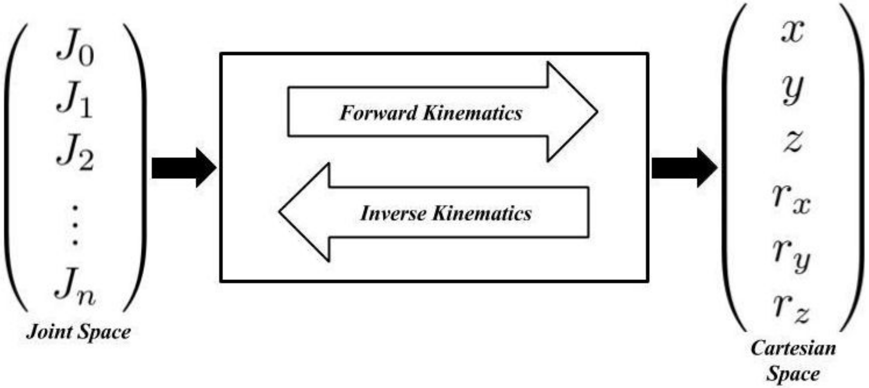




Read: Frame rotations and representations

Forward Kinematics vs Inverse Kinematics





Spatial Representations

⇒ **Axis:** *X, Y, Z*

⇒ **Unite Vectors** are represented with ***hats***

- I.E.: $\hat{Y}$ is the unit vector along *Y* axis

⇒ **{*A*}** represents a frame of reference named **A**

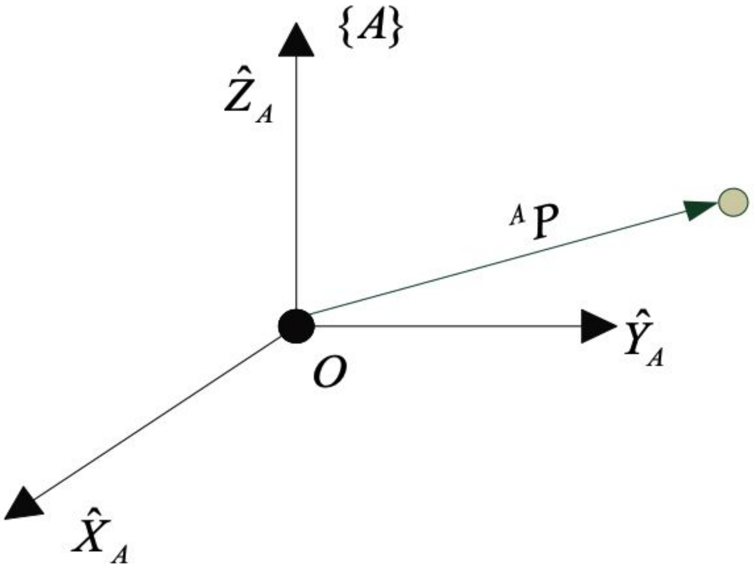
⇒ ^{A}P means a point **P** represented in frame **{A}**

⇒ Global frame of world frame

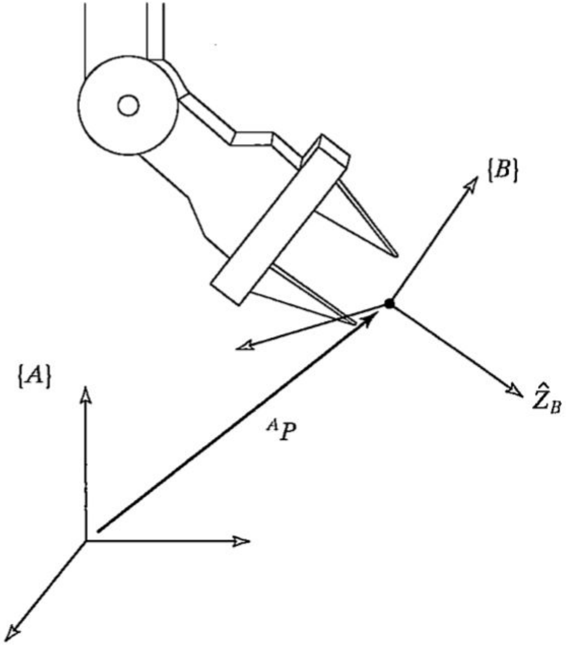
- ***{G}* or *{W}***

How to represent a point ^{B}P *in a different frame, say* ***{A}?***

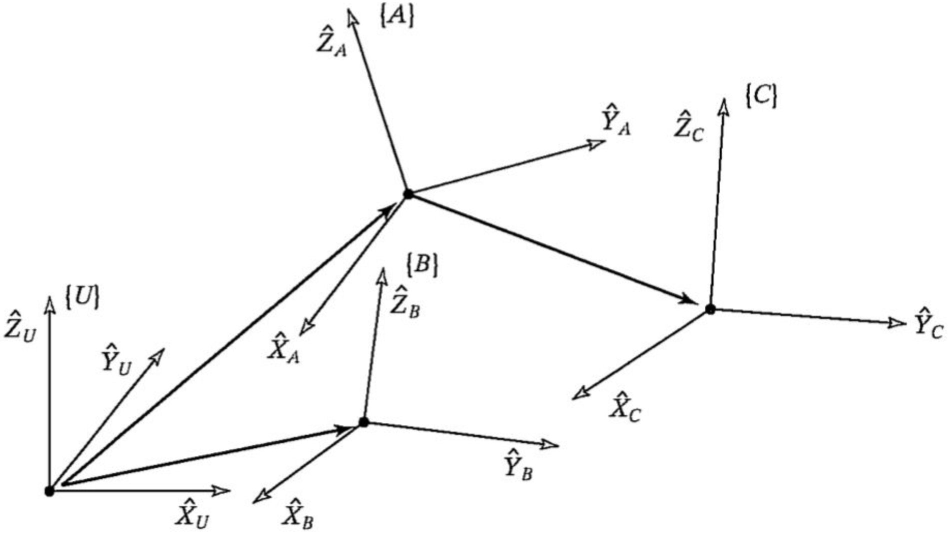
$$^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$



Translation and Orientation



- ⇒ **Translation:** linear displacement
- ⇒ **Rotation:** angular displacement

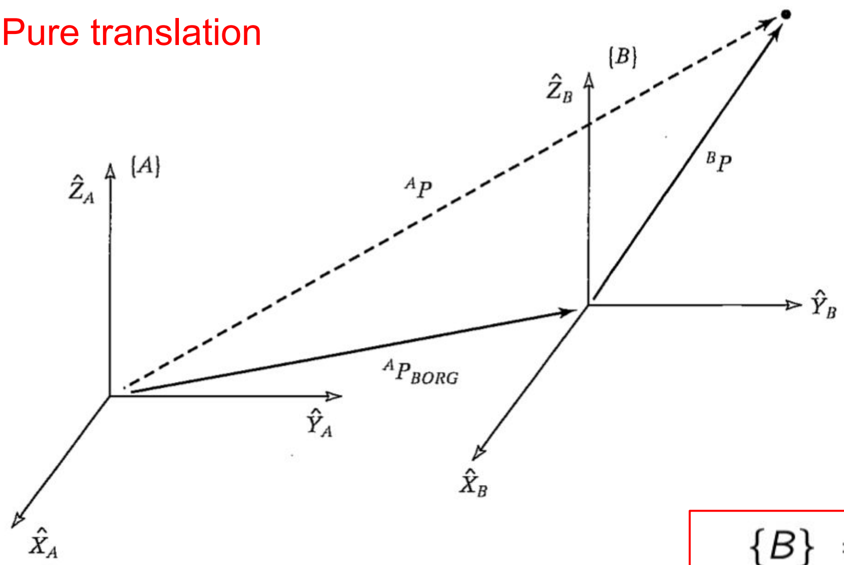


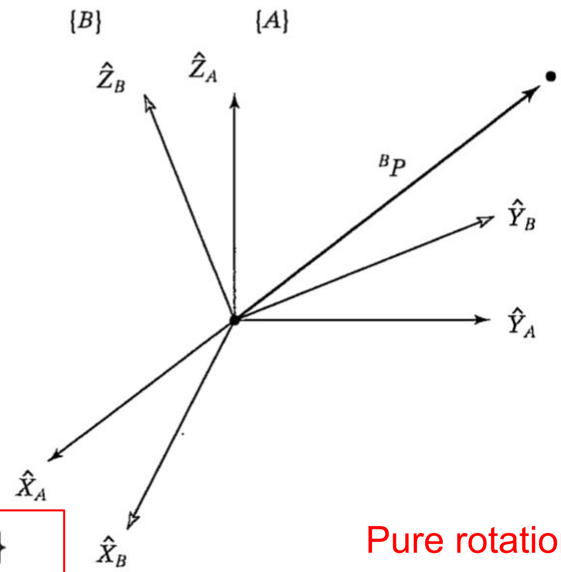
How to represent a point ***^{B}P*** *in a different frame, say* ***{A}?***

If we find pairwise solution, then we can solve complex ones too!

From ^BP To ^AP

Pure translation





Pure rotation

$$\{B\} = \{^A_BR, \, ^AP_{Borg}\}$$

$$^AP = ^A_BR \, ^BP + ^AP_{Borg}$$

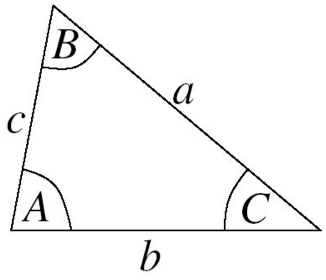
$$^AP = ^BP + ^AP_{Borg}$$

*Trivial case: example**?***

$$^AP = ^A_BR \, ^BP$$

How?

Preliminaries: Sine And Cosine rules



Sine Rule

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \quad \text{or} \quad \frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

(for finding sides) (for finding angles)

Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos(A) \quad \text{or} \quad \cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

(for finding sides) (for finding angles)

sin(A-B)= sinAcosB-cosAsinB

sin(A+B)= sinAcosB+cosAsinB

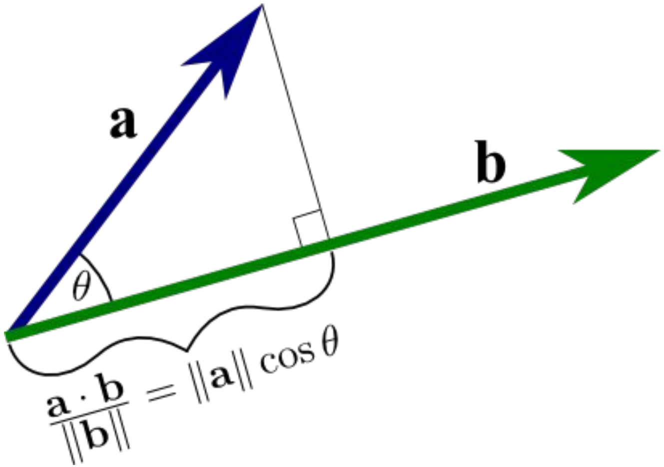
cos(A+B)=cosAcosB-sinAsinB

cos(A-B)=cosAcosB+sinAsinB

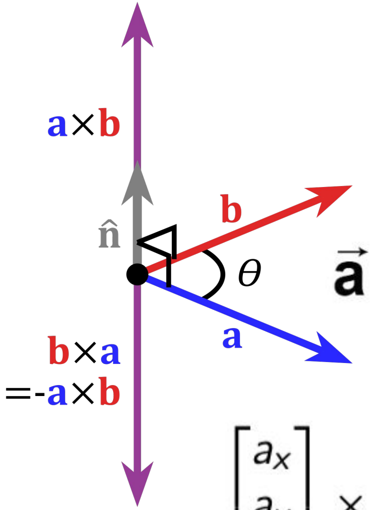
Short casual notation!
***C** (A+B) = **C**A CB - **S**A SB*
***C** (A-B) = **C**A CB + **S**A SB*

*2 **CA CB** = **C**(A+B) + **C**(A-B)*
*2 **SA SB** = **C**(A-B) - **C**(A+B)*

Preliminaries: Dot And Cross Product



$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

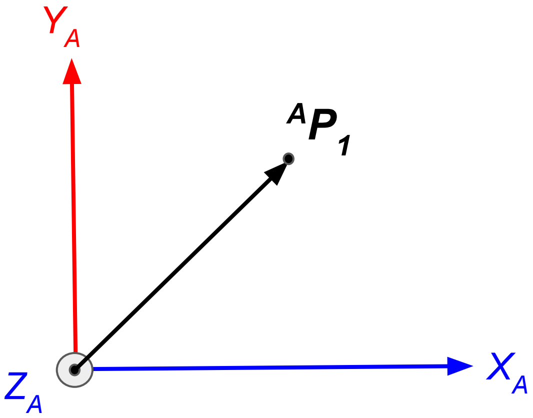


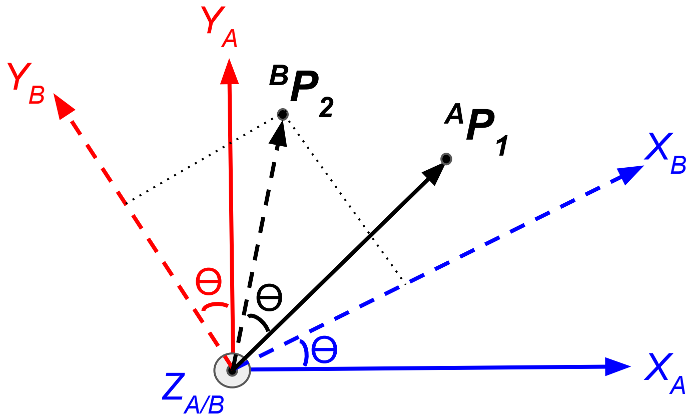
$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\,\sin\theta\,\hat{n}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z\ b_y \\ a_x b_z - a_z\ b_x \\ a_x b_y - a_y\ b_x \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \cdot \mathbf{b} = \underbrace{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}}_{[\mathbf{a}]_\times} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Rotation Around Z Axis

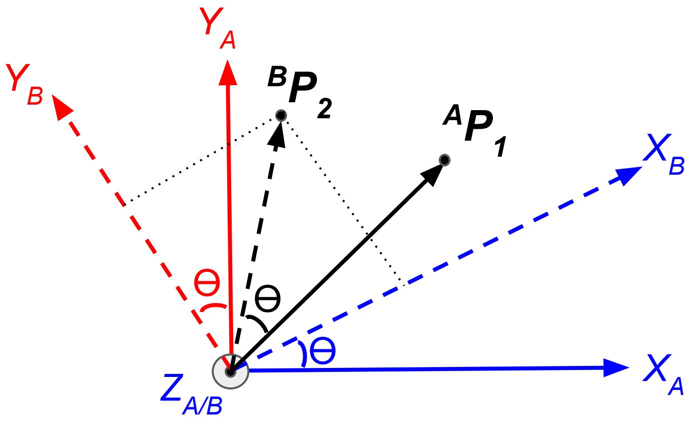




$$^B P_2 = {}^A P_1$$

$$^A P_2 = {}^A_B R \, {}^B P_2 = {}^A_B R \, {}^A P_1$$

Rotation Around Z Axis



$$^BP_2 = {}^AP_1$$

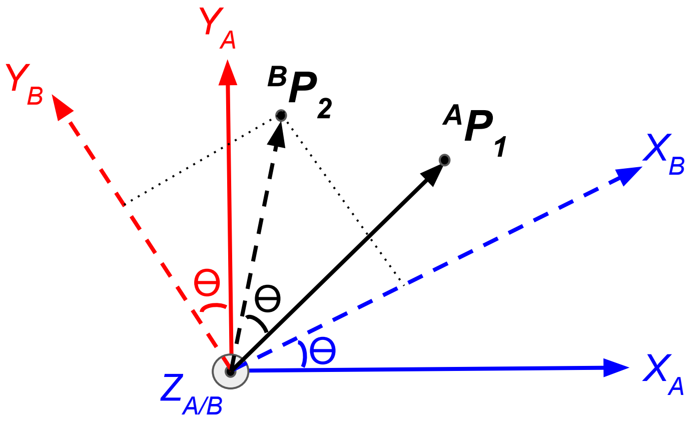
$$^AP_2 = {}^A_BR\,{}^BP_2 = {}^A_BR\,{}^AP_1$$

$$^A_BR = \left[{}^A\hat{X}_B \; {}^A\hat{Y}_B \; {}^A\hat{Z}_B\right] = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}$$

Proof:

$$^AP_2 = \begin{bmatrix} {}^BP_2 \cdot {}^B\hat{X}_A \\ {}^BP_2 \cdot {}^B\hat{Y}_A \\ {}^BP_2 \cdot {}^B\hat{Z}_A \end{bmatrix} = \begin{bmatrix} {}^B\hat{X}'_A \cdot {}^BP_2 \\ {}^B\hat{Y}'_A \cdot {}^BP_2 \\ {}^B\hat{Z}'_A \cdot {}^BP_2 \end{bmatrix} = \left[{}^A\hat{X}_B \; {}^A\hat{Y}_B \; {}^A\hat{Z}_B\right] \; {}^BP_2 = \left[{}^A\hat{X}_B \; {}^A\hat{Y}_B \; {}^A\hat{Z}_B\right] \; {}^AP_1$$

Rotation Around Z Axis



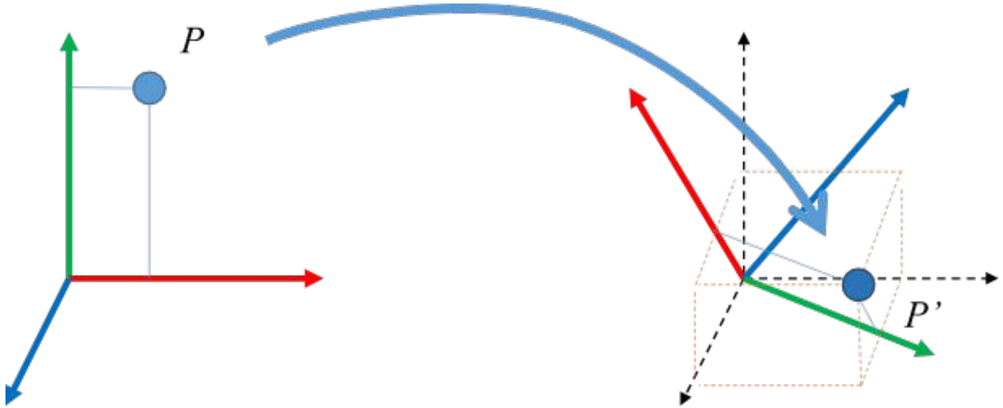
$$^A_BR = [^A\hat{X}_B \; ^A\hat{Y}_B \; ^A\hat{Z}_B] = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}$$

$$^A_BR = R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_y\,(\Theta)$= ? and $R_z\,(\Theta)$= ?

Elementary Rotations: X / Y / Z

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\boldsymbol{p} = (x, y, z)$$

$$\boldsymbol{p}' = R\boldsymbol{p} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Properties Of A Rotation Matrix

$$ {}^A_BR = [{}^A\hat{X}_B \ {}^A\hat{Y}_B \ {}^A\hat{Z}_B] = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} $$

$$ {}^A_BR = [{}^A\hat{X}_B \ {}^A\hat{Y}_B \ {}^A\hat{Z}_B] = \begin{bmatrix} {}^B\hat{X}_A^T \\ {}^B\hat{Y}_A^T \\ {}^B\hat{Z}_A^T \end{bmatrix} $$

$$ {}^A_BR^T \cdot {}^A_BR = [{}^A\hat{X}_B \ {}^A\hat{Y}_B \ {}^A\hat{Z}_B] \cdot \begin{bmatrix} {}^B\hat{X}_A^T \\ {}^B\hat{Y}_A^T \\ {}^B\hat{Z}_A^T \end{bmatrix} = \mathbf{I}_3 $$

⇒ Det(R) = 1

⇒ $R^T = R^{-1}$ and $R^TR = I_{3x3}$

See https://en.wikipedia.org/wiki/Rotation_matrix

⇒ <u>Learn about:</u>

- SO(3), Li algebra, Li group
- Eigenvalues of rotation matrix
- Clockwise / anti-clockwise rotation

In-class Practice

$$ {}^A_BR = [{}^A\hat{X}_B \ {}^A\hat{Y}_B \ {}^A\hat{Z}_B] = \begin{bmatrix} \hat{X}_B\cdot\hat{X}_A & \hat{Y}_B\cdot\hat{X}_A & \hat{Z}_B\cdot\hat{X}_A \\ \hat{X}_B\cdot\hat{Y}_A & \hat{Y}_B\cdot\hat{Y}_A & \hat{Z}_B\cdot\hat{Y}_A \\ \hat{X}_B\cdot\hat{Z}_A & \hat{Y}_B\cdot\hat{Z}_A & \hat{Z}_B\cdot\hat{Z}_A \end{bmatrix} $$

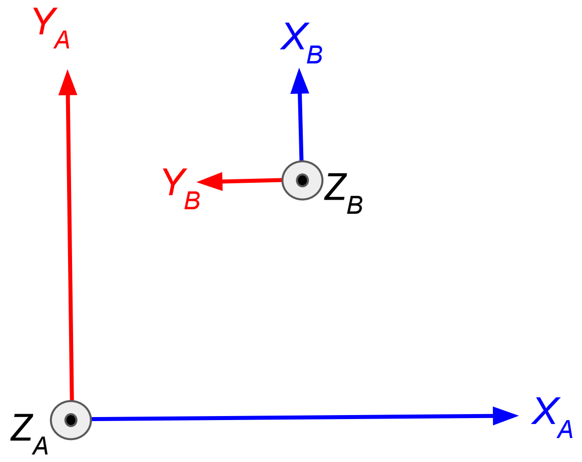
$$ \hat{Z}_A||\hat{Z}_B, \quad so \quad {}^A\hat{Z}_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

$$ {}^A\hat{X}_B = A\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} $$

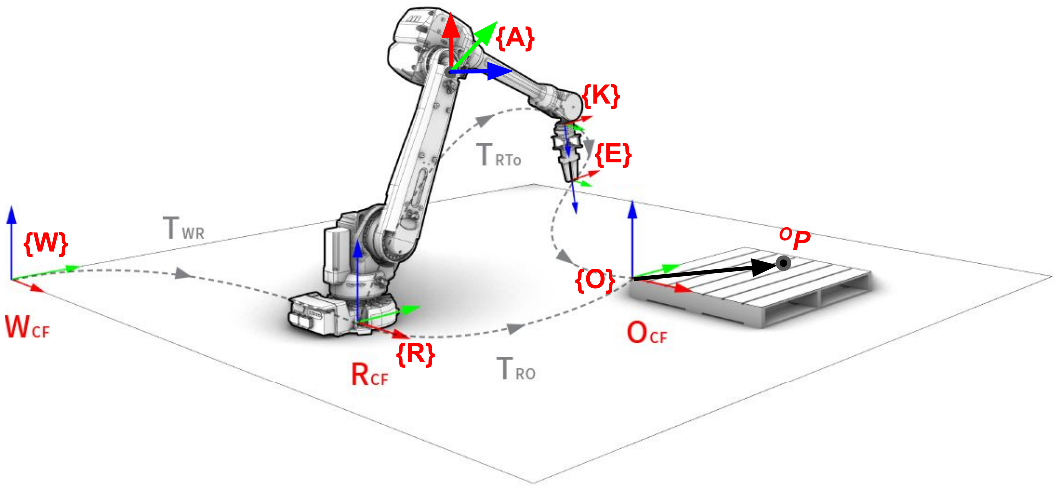
$$ {}^A\hat{Y}_B = A\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^A_BR = [{}^A\hat{X}_B \ {}^A\hat{Y}_B \ {}^A\hat{Z}_B] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^A_BR = \begin{bmatrix} \hat{X}_B\cdot\hat{X}_A & \hat{Y}_B\cdot\hat{X}_A & \hat{Z}_B\cdot\hat{X}_A \\ \hat{X}_B\cdot\hat{Y}_A & \hat{Y}_B\cdot\hat{Y}_A & \hat{Z}_B\cdot\hat{Y}_A \\ \hat{X}_B\cdot\hat{Z}_A & \hat{Y}_B\cdot\hat{Z}_A & \hat{Z}_B\cdot\hat{Z}_A \end{bmatrix} = \begin{bmatrix} C(\frac{\pi}{2}) & -S(\frac{\pi}{2}) & 0 \\ S(\frac{\pi}{2}) & C(\frac{\pi}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



Practice: Find *P* In Every Frame!



$$^{E}P = {}^{E}P_{Oorg} + {}^{E}_{O}R\,{}^{O}P$$

$$^{K}P = {}^{K}P_{Eorg} + {}^{K}_{E}R\,{}^{E}P$$

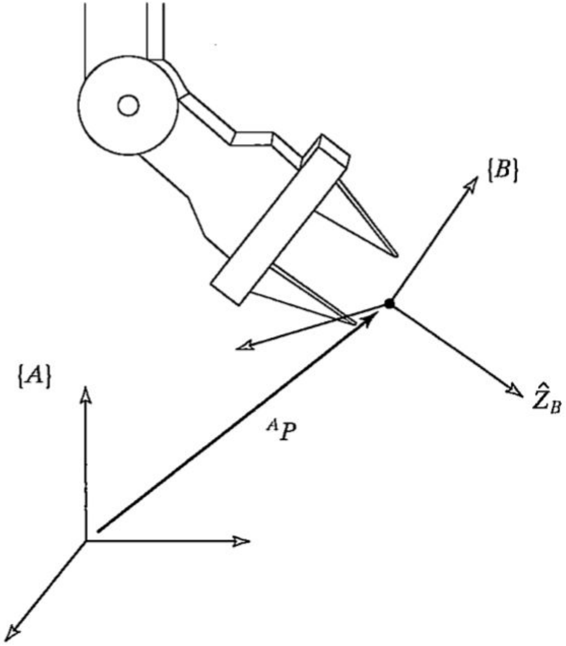
$$^{A}P = {}^{A}P_{Korg} + {}^{A}_{K}R\,{}^{K}P$$

$^{R}P = ?$

$^{W}P = ?$

How to do all these calculations more efficiently?

^{B}P To ^{A}P : Homogeneous Representation



$$^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{Borg}$$

$$^{A}P = {}^{A}_{B}\mathbf{T}\,{}^{B}P$$

$$\begin{bmatrix} ^{A}P \\ 1 \end{bmatrix} = \underbrace{\left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}_{{}^{A}_{B}T} \begin{bmatrix} ^{B}P \\ 1 \end{bmatrix}$$

Special cases

- Pure Translation: $\mathbf{R} = \mathbf{I}_{3x3}$
- Pure Rotation: $^{A}\boldsymbol{P}_{BORG} = \mathbf{0}_{3x1}$

In-class Practice

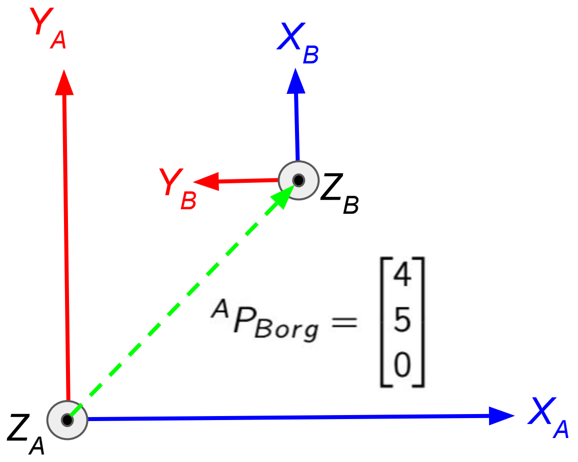
$$ {}^A_BR = [{}^A\hat{X}_B \ {}^A\hat{Y}_B \ {}^A\hat{Z}_B] = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} $$

$$ \hat{Z}_A || \hat{Z}_B, \quad so \quad {}^A\hat{Z}_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

$$ {}^A\hat{X}_B = A\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} $$

$$ {}^A\hat{Y}_B = A\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^A_BR = [{}^A\hat{X}_B \ {}^A\hat{Y}_B \ {}^A\hat{Z}_B] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



$$ {}^AP_{Borg} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} $$

$$ {}^A_B\mathbf{T} = \left[\begin{array}{c|c} {}^A_BR & {}^AP_{Borg} \\ \hline 0 \ \ 0 \ \ 0 & 1 \end{array} \right] = \begin{bmatrix} 0 & -1 & 0 & 4 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

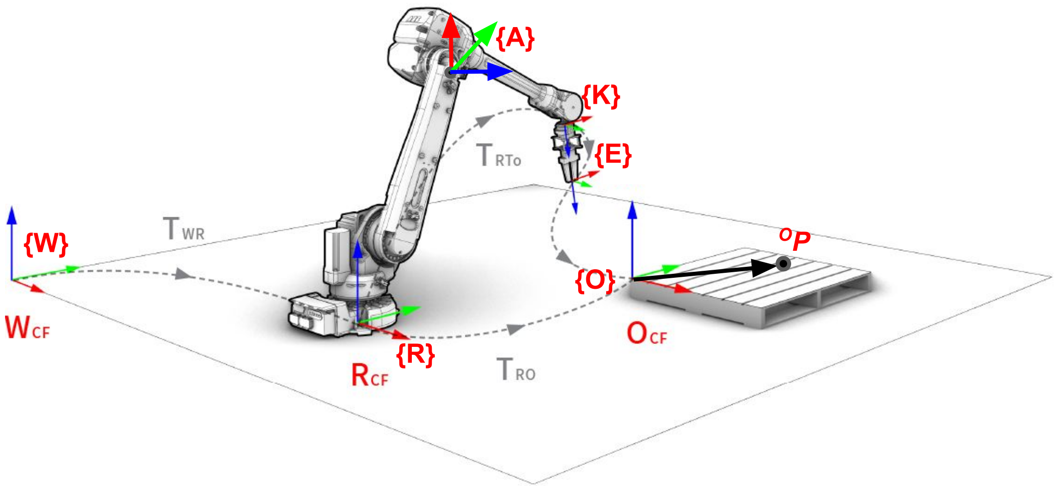
In-class Practice

$$ {}^{A}_{B}\mathbf{T} = \begin{bmatrix} 0 & -1 & 0 & -5 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- Find ${}^{A}_{B}R$ and ${}^{A}P_{Borg}$
- Find ${}^{B}_{A}R$ and ${}^{B}P_{Aorg}$
- Find ${}^{A}P$ if ${}^{B}P = [5 \quad -2 \quad 8]^T$
- Find ${}^{B}_{A}\mathbf{T}$

Can you visualize this?

Practice: Find *P* In Every Frame!



$$ {}^{K}P = {}^{K}_{E}T\,{}^{E}_{O}T\,{}^{O}P = {}^{K}_{O}T\,{}^{O}P $$

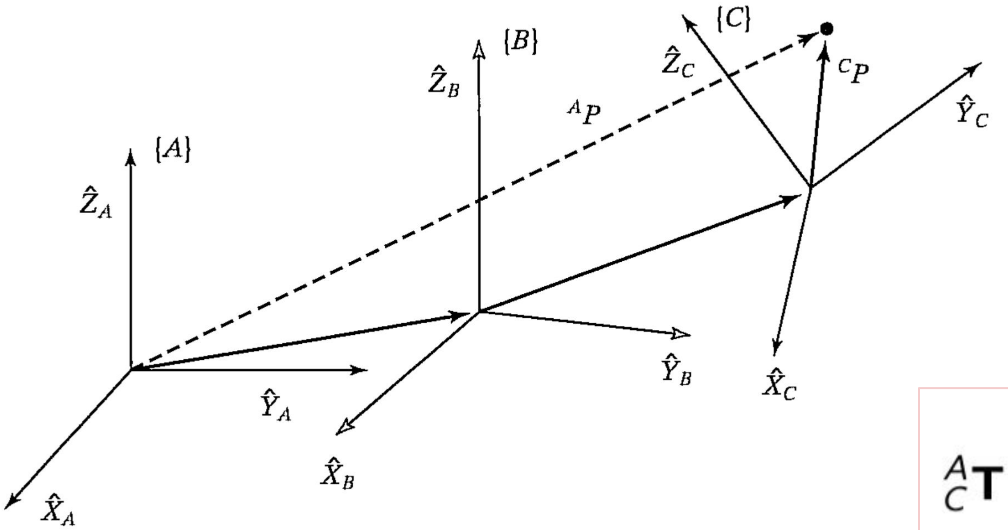
$$ {}^{A}P = {}^{A}_{K}T\,{}^{K}_{E}T\,{}^{E}_{O}T\,{}^{O}P = {}^{A}_{O}T\,{}^{O}P $$

Find ${}^{W}P$

$$ \begin{bmatrix} {}^{E}P \\ 1 \end{bmatrix} = {}^{E}_{O}R\,{}^{O}P + {}^{E}P_{Oorg} = \left[\begin{array}{ccc|c} & {}^{E}_{O}R & & {}^{E}P_{Oorg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{O}P \\ 1 \end{bmatrix} $$

$$ {}^{E}P = {}^{E}_{O}T\,{}^{O}P $$

Compound Transforms



$$^{A}P = {}^{A}_{B}\mathbf{T} \cdot {}^{B}_{C}\mathbf{T} \cdot {}^{C}P$$

$$^{A}_{C}\mathbf{T} = {}^{A}_{B}\mathbf{T} \cdot {}^{B}_{C}\mathbf{T}$$

Proof:

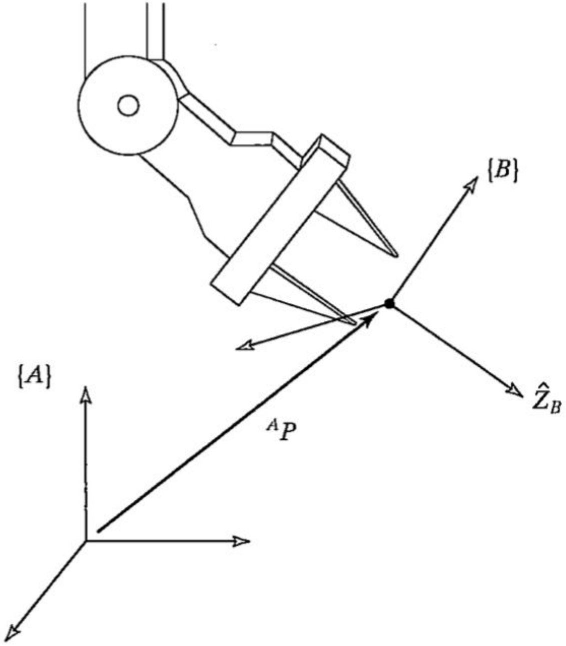
$$^{A}_{B}\mathbf{T} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$^{B}_{C}\mathbf{T} = \left[\begin{array}{ccc|c} & {}^{B}_{C}R & & {}^{B}P_{Corg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$^{A}_{C}\mathbf{T} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R \cdot {}^{B}_{C}R & & {}^{A}_{B}R \cdot {}^{B}P_{Corg} + {}^{A}P_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$^{A}_{C}\mathbf{T} \equiv \{{}^{A}_{C}R\ ,\ {}^{A}P_{Corg}\} = \{{}^{A}_{B}R \cdot {}^{B}_{C}R,\ \ {}^{A}_{B}R \cdot {}^{B}P_{Corg} + {}^{A}P_{Borg}\}$$

Inverse Transforms



$$^AP = {}^A_B R \, {}^BP + {}^AP_{Borg}$$

$$^AP = {}^A_B\mathbf{T} \, {}^BP$$

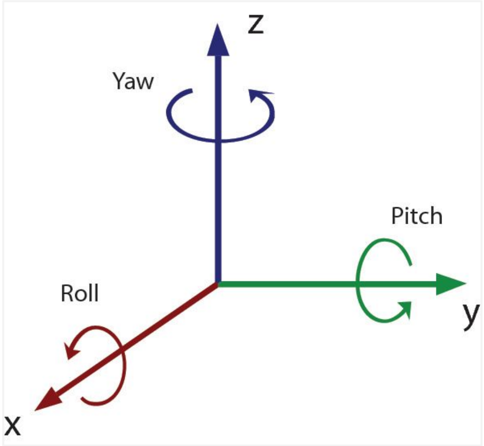
$$\begin{bmatrix} ^AP \\ 1 \end{bmatrix} = \underbrace{\left[\begin{array}{ccc|c} & {}^A_BR & & {}^AP_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}_{{}^A_B\mathrm{T}} \begin{bmatrix} ^BP \\ 1 \end{bmatrix}$$

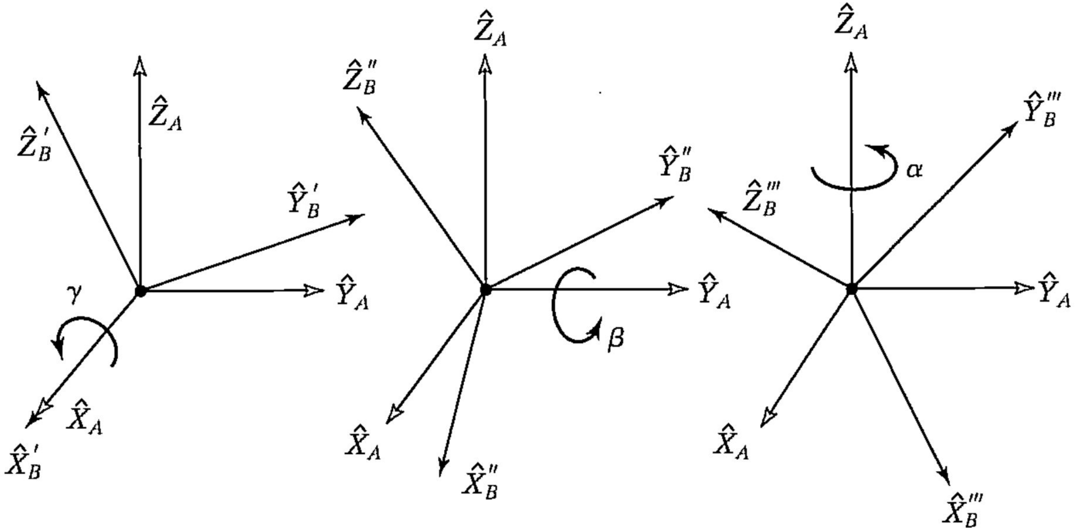
$${}^B_A\mathbf{T} \equiv \{{}^B_AR, \; {}^BP_{Aorg}\} = {}^A_B\mathbf{T}^{-1}$$

Derivation:

$${}^B_A\mathbf{T} = \left[\begin{array}{ccc|c} & {}^A_BR^T & & -{}^A_BR^T \cdot {}^AP_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

X-Y-Z Fixed Angle Rotation





$R_{X_A}(\gamma)$ → $R_{Y_A}(\beta)$ → $R_{Z_A}(\alpha)$

Rotate ***{B}*** *around* $\boldsymbol{X_A}$ *(ɣ) then around* $\boldsymbol{Y_A}$*(β), then around* $\boldsymbol{Z_A}$*(α)*

R(X-Y-Z) Fixed Angle

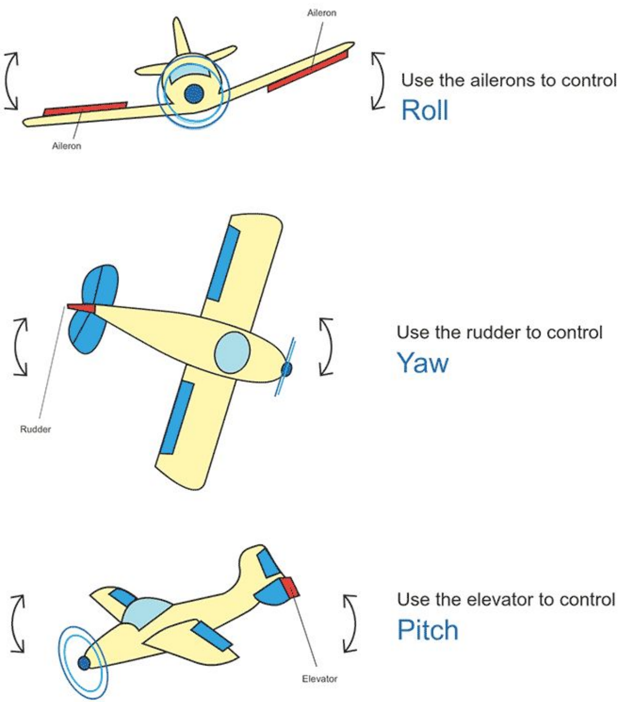
$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)\cdot R_Y(\beta)\cdot R_X(\gamma)
$$

$$
= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}
$$

$$
= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
$$

Rotate ***{B}*** *around* X_A *(ɤ) then around* Y_A*(β), then around* Z_A*(α)*

Solve for α, β, ɤ



R(X-Y-Z) Fixed Angle Solution

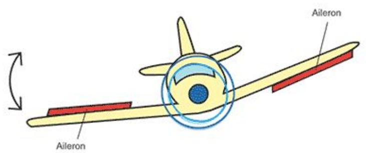
$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)\cdot R_Y(\beta)\cdot R_X(\gamma)
$$

$$
= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
$$

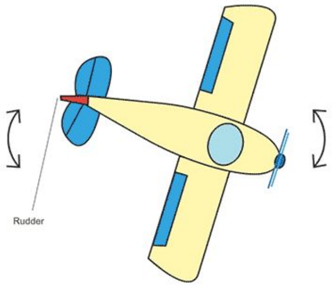
$$
= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

Solve for α, β, γ:

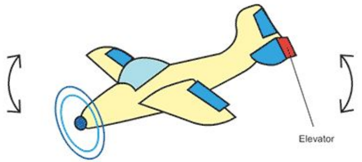
9 equations, 3 unknowns



Use the ailerons to control
Roll



Use the rudder to control
Yaw



Use the elevator to control
Pitch

General Solutions:

$c\beta \neq 0$

$-\pi/2 \leq \beta \leq \pi/2$

$$
\beta = Atan2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})
$$

$$
\alpha = Atan2(\psi\ r_{21}, \psi\ r_{11}), \quad \psi = sign(c\beta)
$$

$$
\gamma = Atan2(\psi\ r_{32}, \psi\ r_{33}), \quad \psi = sign(c\beta)
$$

R(X-Y-Z) Fixed Angle Solution

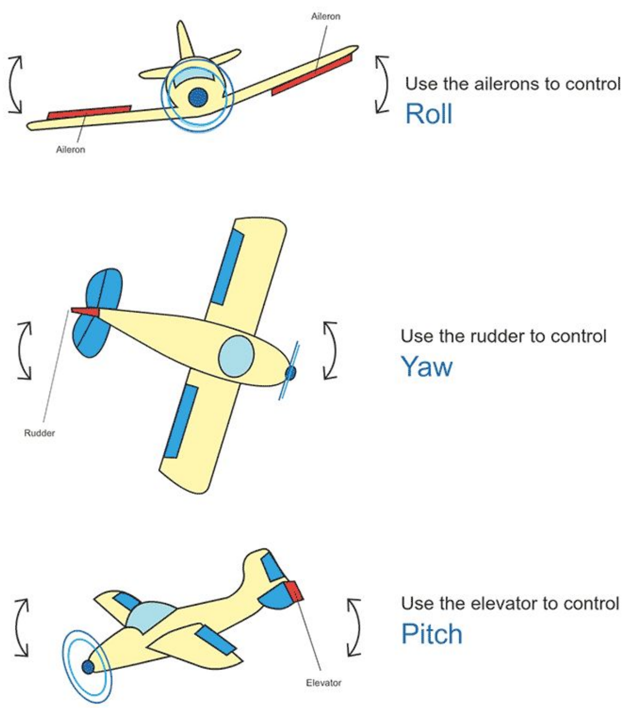
$${}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)\cdot R_Y(\beta)\cdot R_X(\gamma)$$

$$= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Solve for α, β, γ:

9 equations, 3 unknowns



Degenerate Solutions:

$c\beta = 0$

$\beta = \pm \pi/2$

- **Case-I:** $\beta = \frac{\pi}{2}$; $c\beta = 0$; $s\beta = 1$
  - * $\gamma - \alpha = Atan2(r_{12}, r_{22})$
  - * Set $\alpha = 0, \gamma = Atan2(r_{12}, r_{22})$
- **Case-II:** $\beta = -\frac{\pi}{2}$; $c\beta = 0$; $s\beta = -1$
  - * $\gamma + \alpha = Atan2(-r_{12}, r_{22})$
  - * Set $\alpha = 0, \gamma = Atan2(-r_{12}, r_{22})$

Euler Angles: Rotation Along Moving Axes

Rotate around ***Z'***$_B$ *(ψ), then*

Rotate around ***Y'***$_B$ *(Θ), then*

Rotate around ***X'***$_B$*(φ).*

$$ {}^{A}_{B}R_{Z'Y'X'}(\Phi, \Theta, \Psi) = R_{Z'}(\Phi) \cdot R_{Y'}(\Theta) \cdot R_{X'}(\Psi) $$

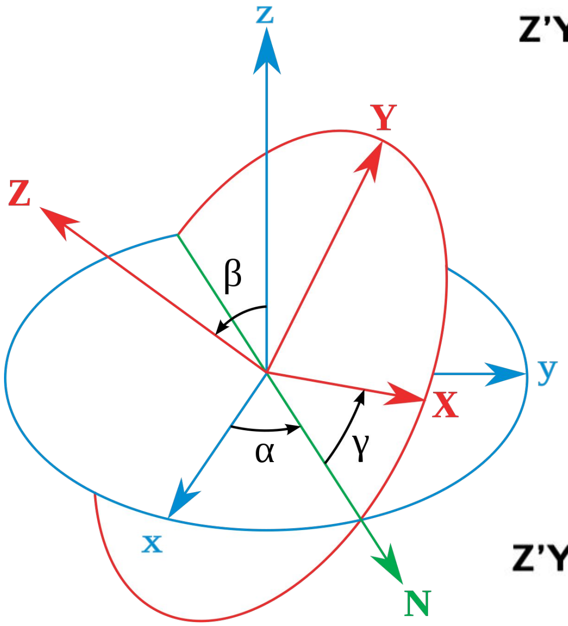
Pay attention to the order!

Why the order is reversed?

Intrinsic rotations are composed in reverse order to extrinsic rotations

See explanation here.

Euler Rotations



Z'Y'X' Euler: $^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_{Z'}(\alpha)\cdot R_{Y'}(\beta)\cdot R_{X'}(\gamma)$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$$= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

Z'Y'Z' Euler: $^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = R_{Z'}(\alpha)\cdot R_{Y'}(\beta)\cdot R_{Z'}(\gamma)$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \cdot \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mixed (Euler-Fixed) Rotations

Rotate around ***Z'$_B$*** *(α), then*

Rotate around ***Y'$_B$*** *(β), then*

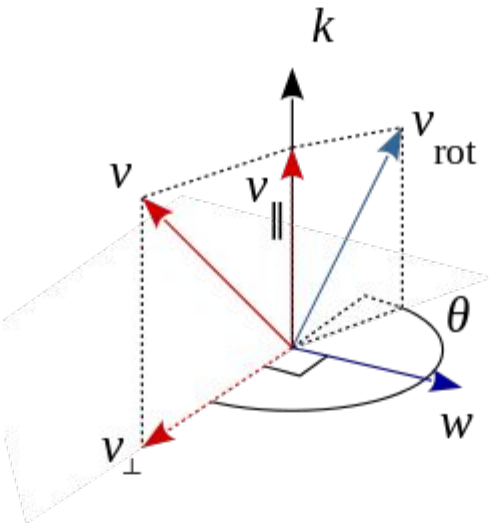
Rotate around ***X$_A$*** *(ɣ).*

$$\textbf{Z'Y'X Mixed:} \quad {}^{A}_{B}R_{Z'Y'X}(\alpha, \beta, \gamma) = R_X(\gamma) \cdot R_{Z'}(\alpha) \cdot R_{Y'}(\beta)$$

Solve for α, β, ɣ given R = [r_{ij}]

- *General case*
- *Degenerate case*

Rodrigues' Rotation Formula



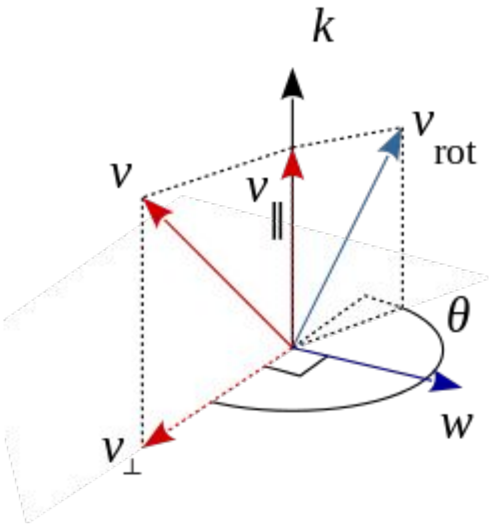
Rodrigues rotation

- An efficient algorithm for computing rotation
- Rotates a vector in space, given
  - An axis of rotation (***k***) and
  - Angle of rotation (***ϴ***)

See more at: [wikipedia]

$$v_{rot} = v\cos\theta + (1-\cos\theta)(k \cdot v)k + (k \times v)\sin\theta$$

Rotation Around Arbitrary Axis



Equivalent (angle, axis) rotation

- Expression of R(***k***, ***ϴ***) in matrix form
  - ***k:*** axis of rotation (unit vector)
  - ***ϴ:*** angle of rotation

See more at: [wikipedia]

$$R(k,\theta) = \begin{bmatrix} k_x^2(1-c\theta)+c\theta & k_x\ k_y(1-c\theta)-k_z s\theta & k_x\ k_z(1-c\theta)+k_y s\theta \\ k_x\ k_y(1-c\theta)+k_z s\theta & k_y^2(1-c\theta)+c\theta & k_y\ k_z(1-c\theta)-k_x s\theta \\ k_x\ k_z(1-c\theta)-k_y s\theta & k_y\ k_z(1-c\theta)+k_x s\theta & k_z^2(1-c\theta)+c\theta \end{bmatrix}$$

Rotation In Quaternions

Unit quaternion: $\mathbf{q} = \begin{bmatrix} \bar{q} \\ q_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}; \quad |q| = 1$

$$\mathbf{q} = q_1 i + q_2 j + q_3 k + q_4$$

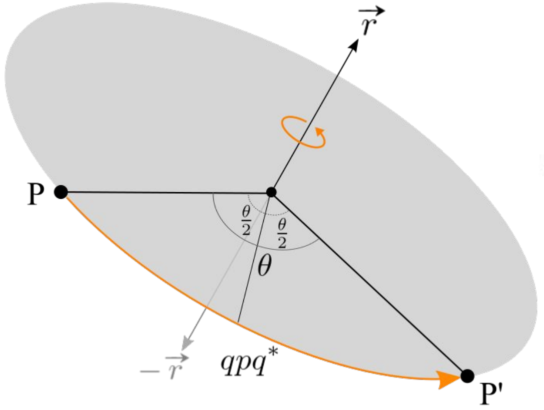
$$i^2 = j^2 = k^2 = ijk = -1$$

Rotation around a unit quaternion

$$\mathbf{q} = \begin{bmatrix} \bar{r}\ s(\theta/2) \\ c(\theta/2) \end{bmatrix} = \begin{bmatrix} r_x\ s(\theta/2) \\ r_y\ s(\theta/2) \\ r_z\ s(\theta/2) \\ c(\theta/2) \end{bmatrix}; \quad |q| = |r| = 1$$

See more at:

- OpenGL Blog Stanford Graphics Notes



$$R_q(r,\theta) = \begin{bmatrix} 1-2q_2^2-2q_3^2 & 2(q_1q_2-q_3q_4) & 2(q_1q_3+q_2q_4) \\ 2(q_1q_2+q_3q_4) & 1-2q_1^2-2q_3^2 & 2(q_2q_3-q_1q_4) \\ 2(q_1q_3-q_2q_4) & 2(q_2q_3+q_1q_4) & 1-2q_1^2-2q_2^2 \end{bmatrix}$$

R and q

$$\mathbf{q} = \begin{bmatrix} \bar{r}\ s(\theta/2) \\ c(\theta/2) \end{bmatrix} = \begin{bmatrix} r_x\ s(\theta/2) \\ r_y\ s(\theta/2) \\ r_z\ s(\theta/2) \\ c(\theta/2) \end{bmatrix} ; \quad |q| = |r| = 1$$

$$R_q(r,\theta) = \begin{bmatrix} 1-2q_2^2-2q_3^2 & 2(q_1q_2-q_3q_4) & 2(q_1q_3+q_2q_4) \\ 2(q_1q_2+q_3q_4) & 1-2q_1^2-2q_3^2 & 2(q_2q_3-q_1q_4) \\ 2(q_1q_3-q_2q_4) & 2(q_2q_3+q_1q_4) & 1-2q_1^2-2q_2^2 \end{bmatrix}$$

https://youtu.be/jTgdKoQv738

$$\mathbf{R}\{[1,0,0,0]^\top\} = \mathbf{I}$$
$$\mathbf{R}\{-\mathbf{q}\} = \mathbf{R}\{\mathbf{q}\}$$
$$\mathbf{R}\{\mathbf{q}^*\} = \mathbf{R}\{\mathbf{q}\}^\top$$
$$\mathbf{R}\{\mathbf{q}_1 \otimes \mathbf{q}_2\} = \mathbf{R}\{\mathbf{q}_1\}\mathbf{R}\{\mathbf{q}_2\}$$