Forward Kinematics

EEL 4930/5934: Autonomous Robots

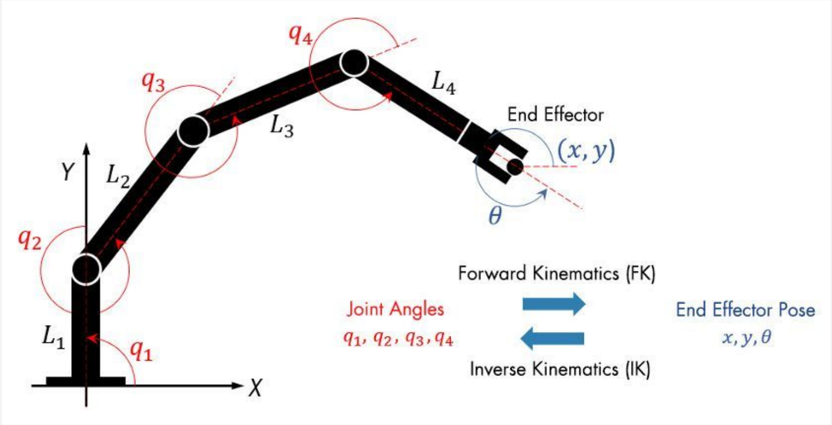
Spring 2023

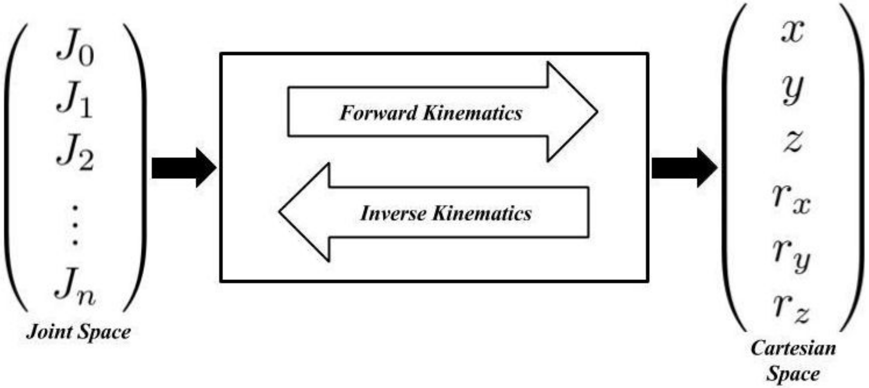
Md Jahidul Islam

Lecture 4

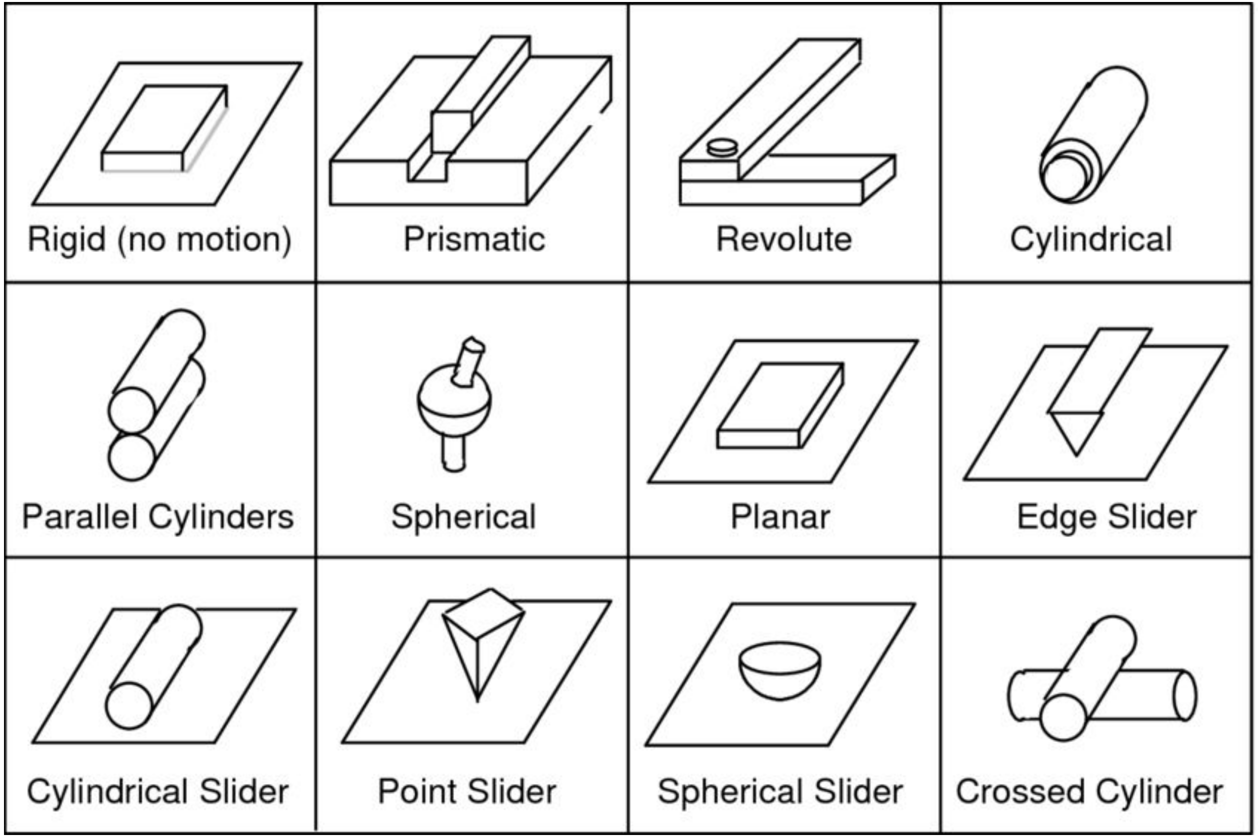
ECE | Florida
Electrical & Computer Engineering
UF | UNIVERSITY of FLORIDA

Forward Kinematics vs Inverse Kinematics

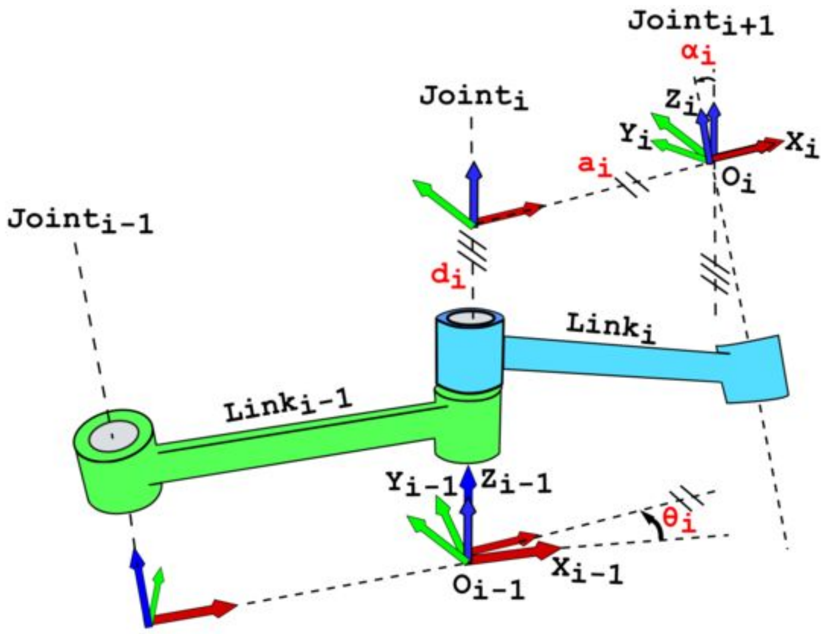




3D Kinematic Joints



Manipulator Kinematics

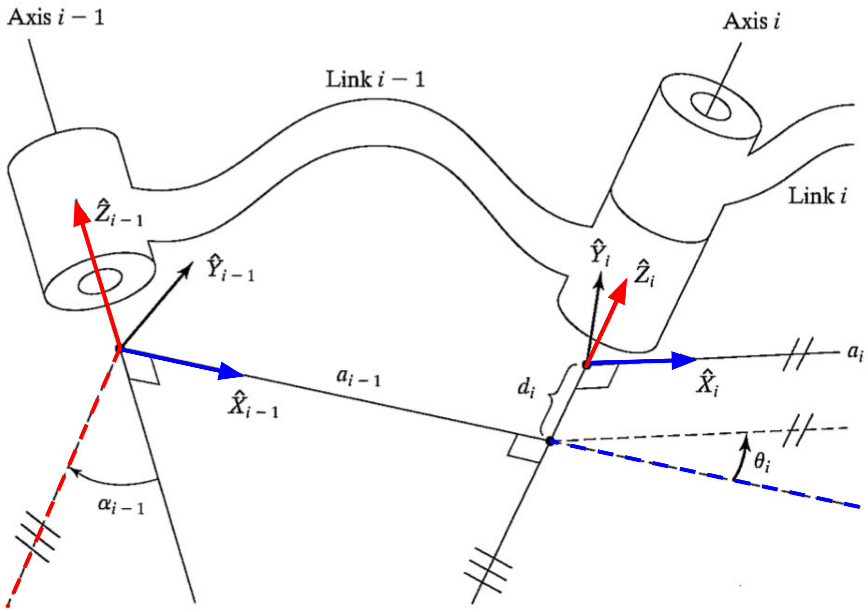


⇒ Denavit-Hartenberg (DH) Notation

- Standardizes kinematic notations
  - Coordinate frames for spatial linkages
  - Solving kinematic motion

- **Four parameters**
  - Link twist: $\boldsymbol{\alpha}$
  - Link length: ***a***
  - Link offset: ***d***
  - Joint angle: $\boldsymbol{\theta}$

DH Notation



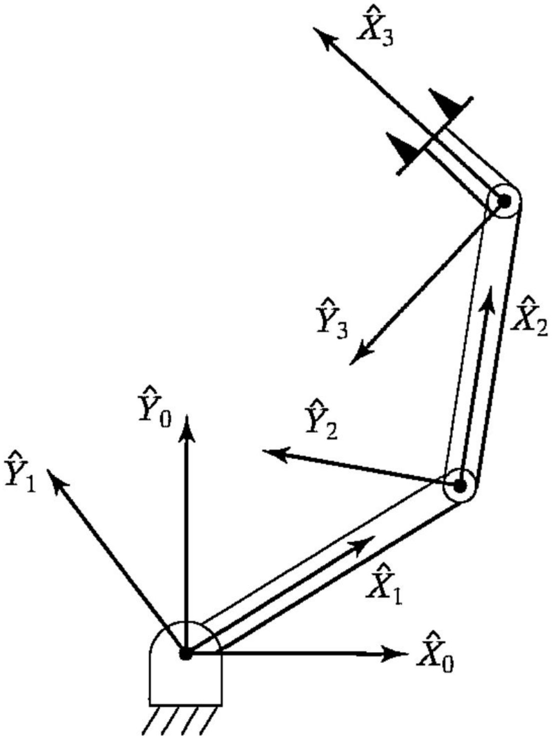
$$\alpha_{i-1} \equiv angle(Z_{i-1}, Z_i)_{X_{i-1}}$$

$$a_{i-1} \equiv distance(Z_{i-1}, Z_i)_{X_{i-1}}$$

$$d_i \equiv distance(X_{i-1}, X_i)_{Z_i}$$

$$\theta_i \equiv angle(X_{i-1}, X_i)_{Z_i}$$

DH Notation: Example 1



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2				
3				

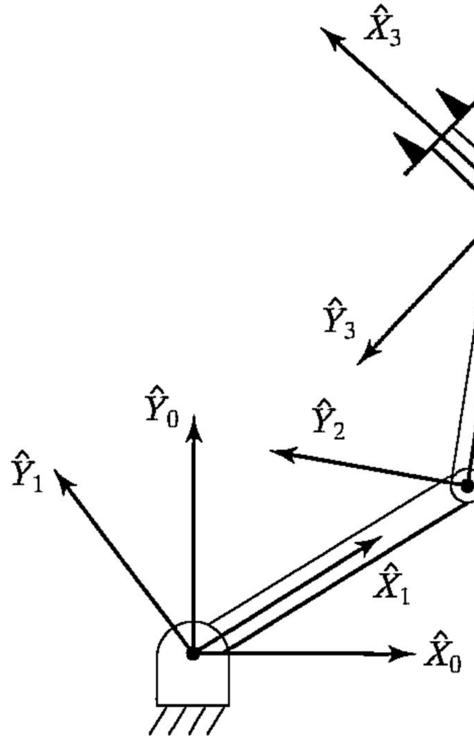
$$\alpha_{i-1} \equiv angle(Z_{i-1}\ ,\ Z_i)_{X_{i-1}}$$

$$a_{i-1} \equiv distance(Z_{i-1}\ ,\ Z_i)_{X_{i-1}}$$

$$d_i \equiv distance(X_{i-1}\ ,\ X_i)_{Z_i}$$

$$\theta_i \equiv angle(X_{i-1}\ ,\ X_i)_{Z_i}$$

DH Notation: Example 1



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	**0**	$\mathbf{L_1}$	**0**	$\boldsymbol{\theta_2}$
3	**0**	$\mathbf{L_2}$	**0**	$\boldsymbol{\theta_3}$

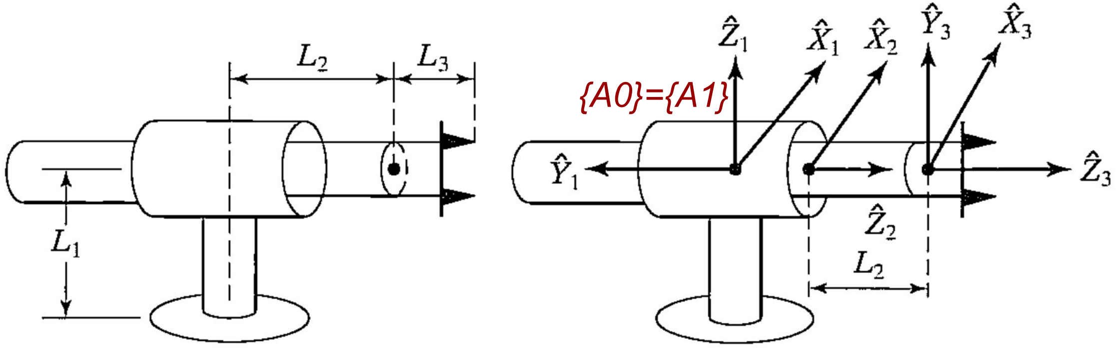
$$\alpha_{i-1} \equiv angle(Z_{i-1}, Z_i)_{X_{i-1}}$$

$$a_{i-1} \equiv distance(Z_{i-1}, Z_i)_{X_{i-1}}$$

$$d_i \equiv distance(X_{i-1}, X_i)_{Z_i}$$

$$\theta_i \equiv angle(X_{i-1}, X_i)_{Z_i}$$

DH Notation: Example 2



$\alpha_{i-1} \equiv angle(Z_{i-1}, Z_i)_{X_{i-1}}$

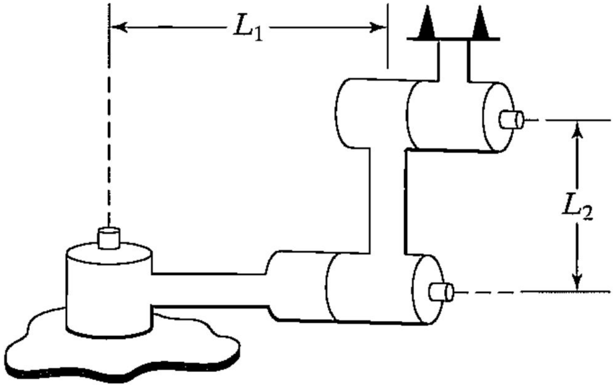
$a_{i-1} \equiv distance(Z_{i-1}, Z_i)_{X_{i-1}}$

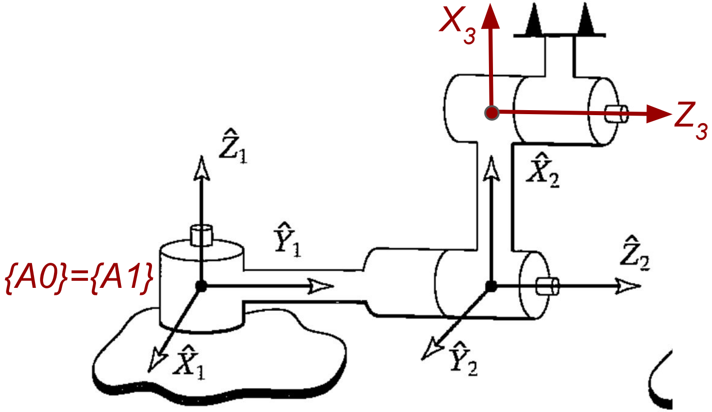
$d_i \equiv distance(X_{i-1}, X_i)_{Z_i}$

$\theta_i \equiv angle(X_{i-1}, X_i)_{Z_i}$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$\pi/2$	0	d_2	0
3	0	0	L_2	θ_3

DH Notation: Example 3





$$\alpha_{i-1} \equiv angle(Z_{i-1}, Z_i)_{X_{i-1}}$$

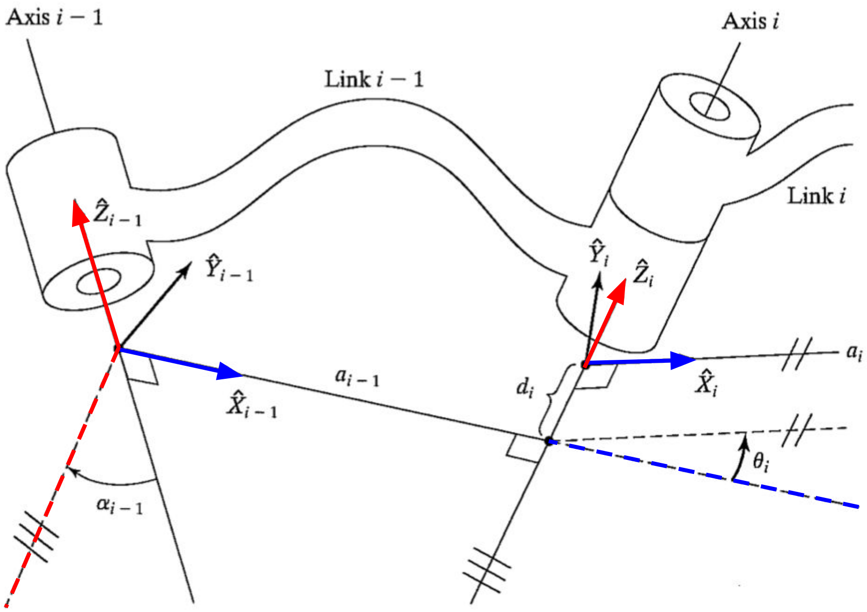
$$a_{i-1} \equiv distance(Z_{i-1}, Z_i)_{X_{i-1}}$$

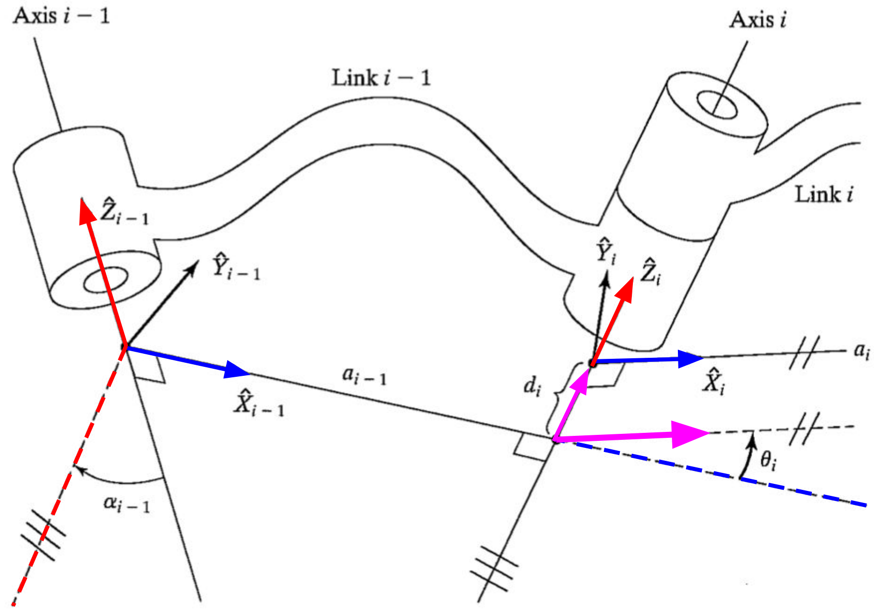
$$d_i \equiv distance(X_{i-1}, X_i)_{Z_i}$$

$$\theta_i \equiv angle(X_{i-1}, X_i)_{Z_i}$$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-\pi/2$	0	L_1	$-\pi/2 + \theta_2$
3	0	L_2	0	θ_3

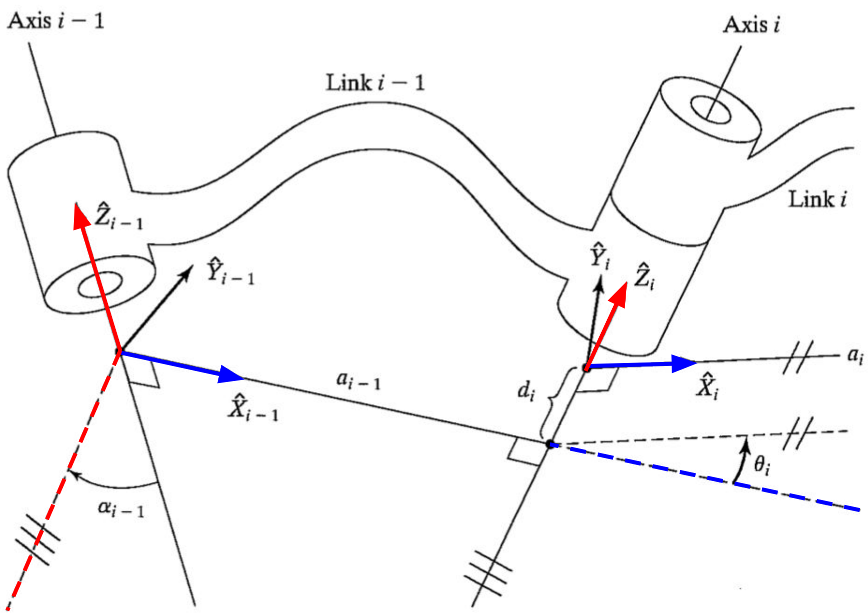
How to Transform: From {i} To {i-1}

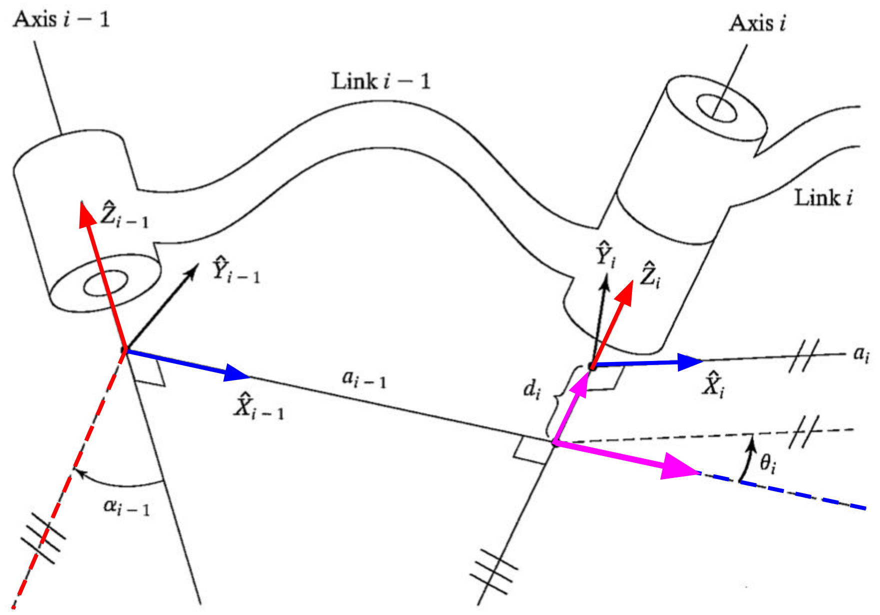




1. Translate $\boldsymbol{d_i}$ *along* $\boldsymbol{Z_i}$	2. Rotate $\boldsymbol{\theta_i}$ *around* $\boldsymbol{Z_i}$
3. Translate $\boldsymbol{a_{i-1}}$ *along* $\boldsymbol{X_{i-1}}$	4. Rotate $\boldsymbol{\alpha_{i-1}}$ *around* $\boldsymbol{X_{i-1}}$

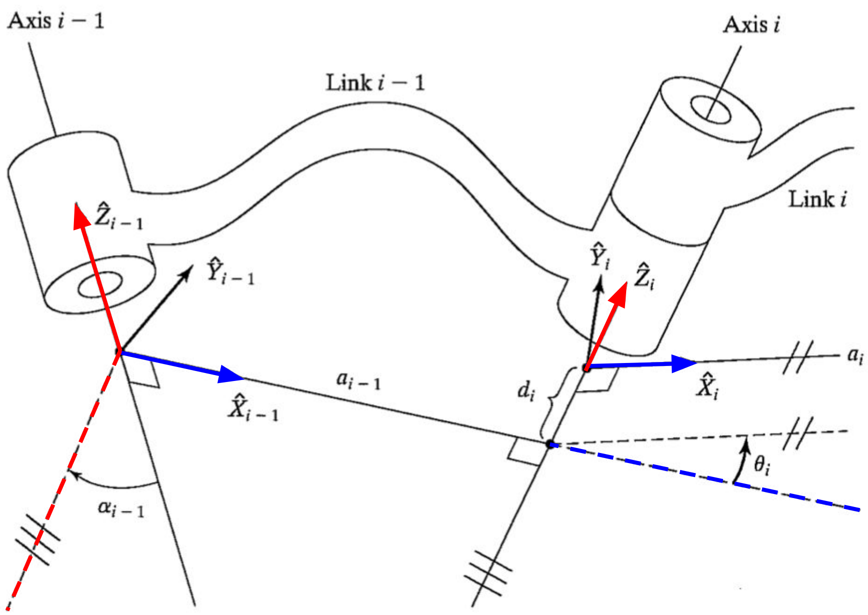
How to Transform: From {i} To {i-1}

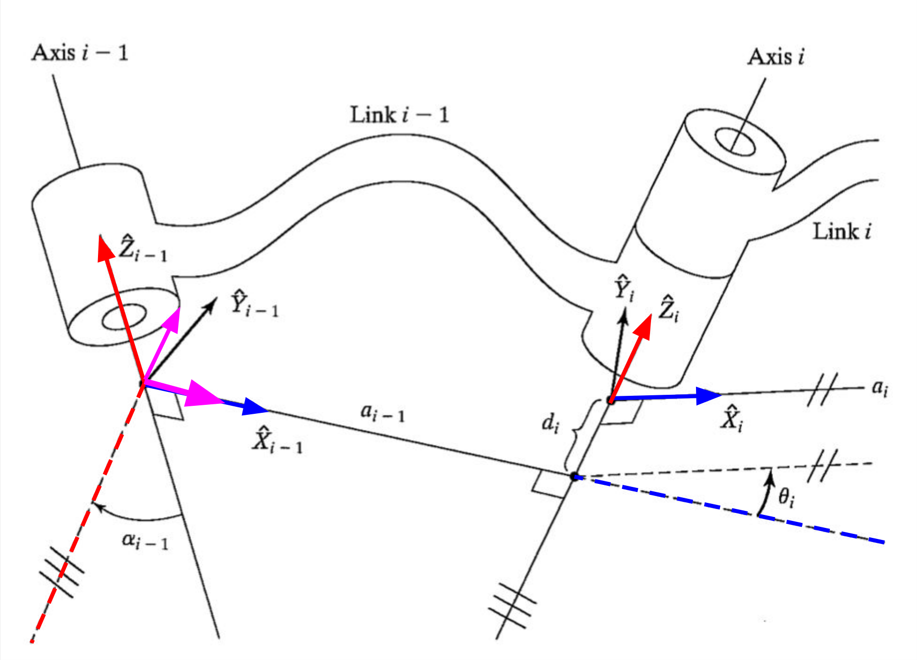




1. Translate $\boldsymbol{d_i}$ *along* $\boldsymbol{Z_i}$	2. Rotate $\boldsymbol{\theta_i}$ *around* $\boldsymbol{Z_i}$
3. Translate $\boldsymbol{a_{i-1}}$ *along* $\boldsymbol{X_{i-1}}$	4. Rotate $\boldsymbol{\alpha_{i-1}}$ *around* $\boldsymbol{X_{i-1}}$

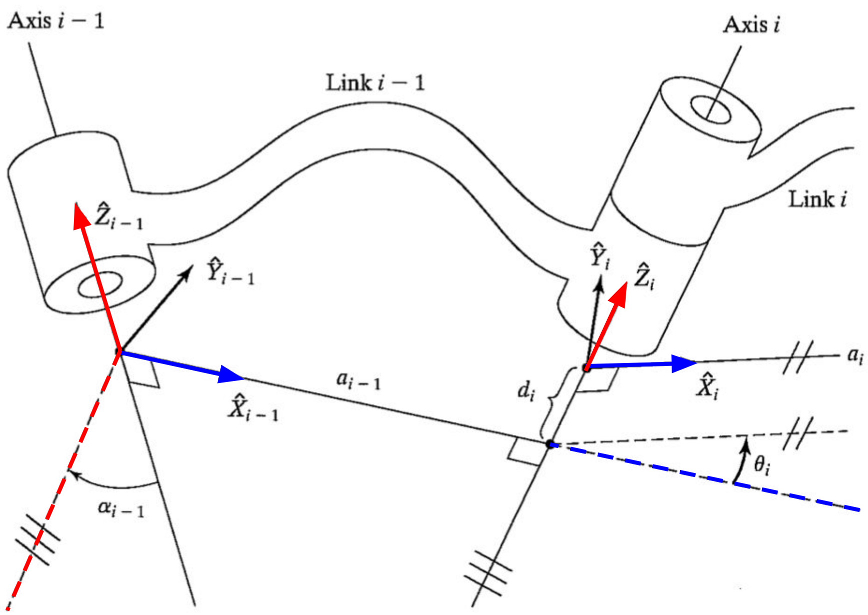
How to Transform: From {i} To {i-1}

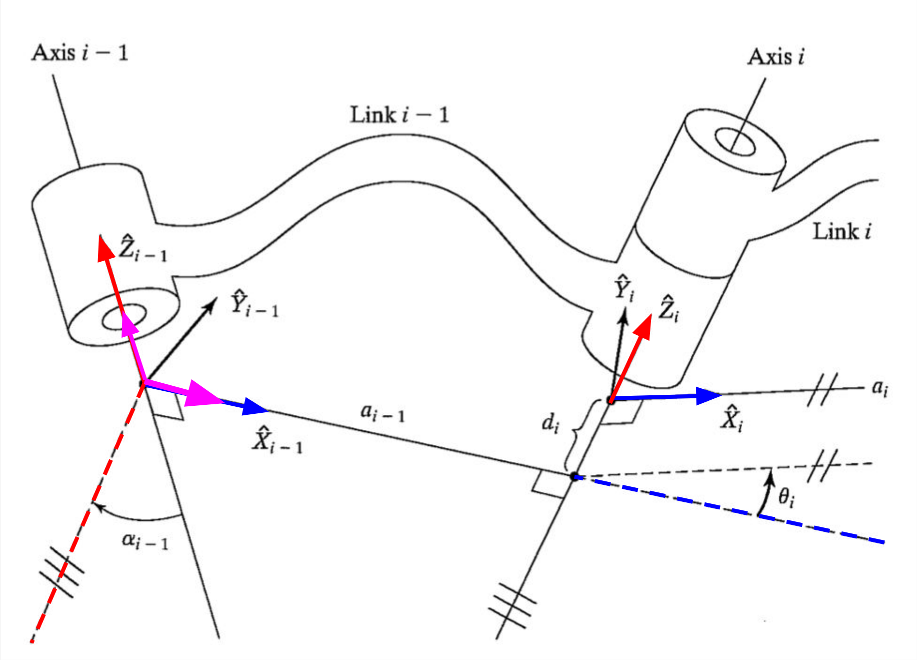




1. Translate d_i along Z_i	2. Rotate θ_i around Z_i
3. Translate a_{i-1} along X_{i-1}	4. Rotate α_{i-1} around X_{i-1}

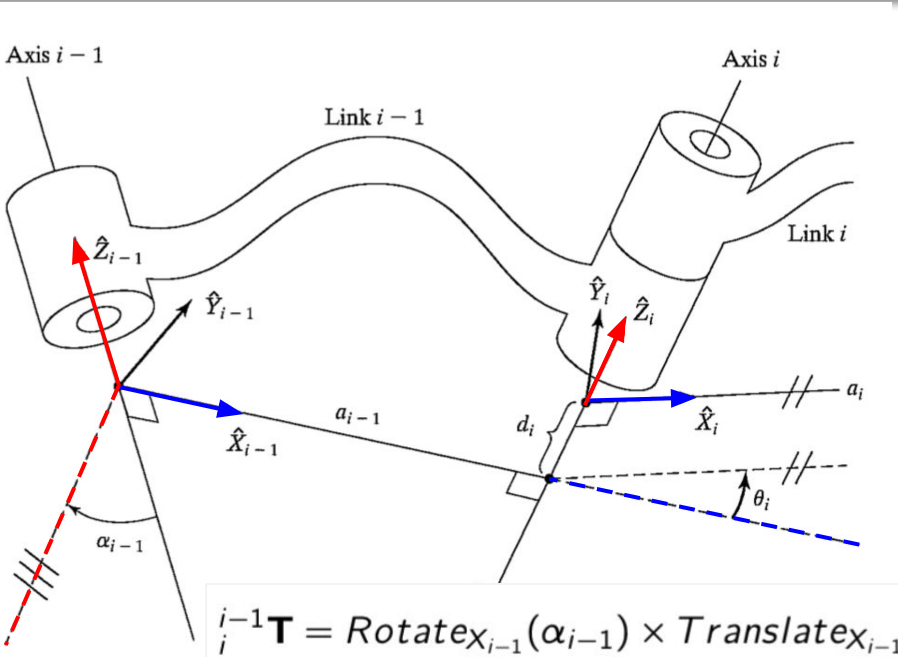
How to Transform: From {i} To {i-1}





1. Translate $\boldsymbol{d_i}$ *along* $\boldsymbol{Z_i}$	2. Rotate $\boldsymbol{\theta_i}$ *around* $\boldsymbol{Z_i}$
3. Translate $\boldsymbol{a_{i-1}}$ *along* $\boldsymbol{X_{i-1}}$	4. Rotate $\boldsymbol{\alpha_{i-1}}$ *around* $\boldsymbol{X_{i-1}}$

Derivation: From {i} To {i-1}

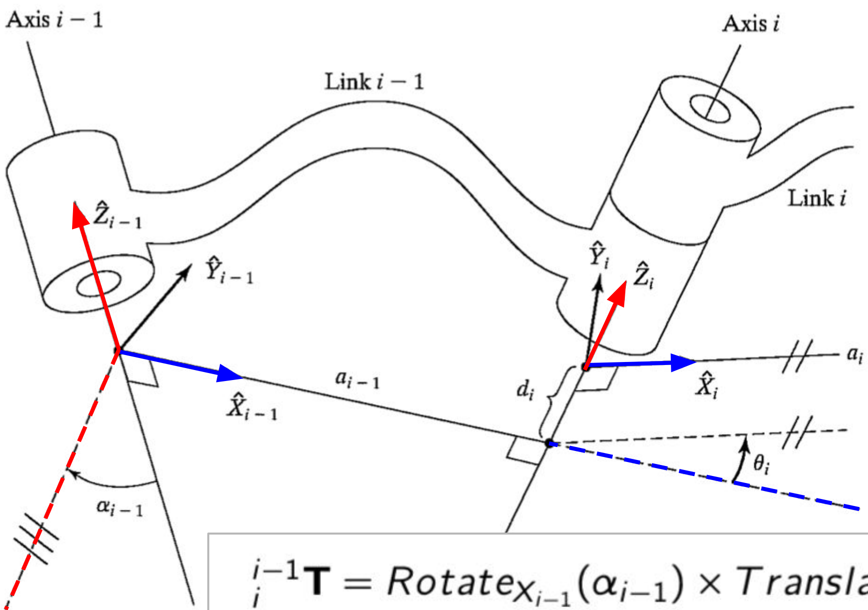


1. Translate d_i along Z_i
2. Rotate θ_i around Z_i
3. Translate a_{i-1} along X_{i-1}
4. Rotate α_{i-1} around X_{i-1}

$$^{i-1}_{i}\mathbf{T} = Rotate_{X_{i-1}}(\alpha_{i-1}) \times Translate_{X_{i-1}}(a_{i-1}) \times Rot_{Z_i}(\theta_i) \times Translate_{Z_i}(d_i)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_{i-1} & -S\alpha_{i-1} & 0 \\ 0 & S\alpha_{i-1} & C\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Derivation: From {i} To {i-1}



1. Translate ***d**$_i$ along **Z**$_i$*

2. Rotate ***θ**$_i$ around **Z**$_i$*

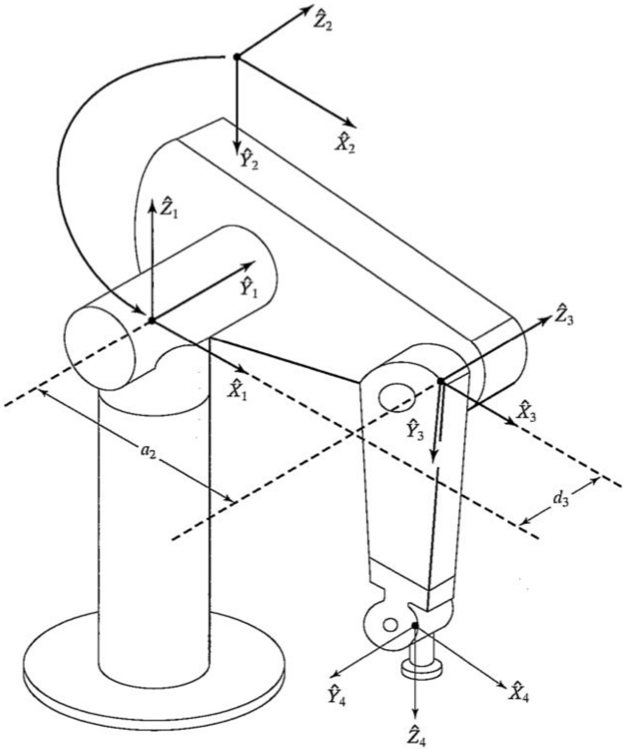
3. Translate ***a**$_{i-1}$ along **X**$_{i-1}$*

4. Rotate ***α**$_{i-1}$ around **X**$_{i-1}$*

$$
{}^{i-1}_{i}\mathbf{T} = Rotate_{X_{i-1}}(\alpha_{i-1}) \times Translate_{X_{i-1}}(a_{i-1}) \times Rot_{Z_i}(\theta_i) \times Translate_{Z_i}(d_i)
$$

$$
= \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{i-1} \\ C\alpha_{i-1} \cdot S\theta_i & C\alpha_{i-1} \cdot C\theta_i & -S\alpha_{i-1} & -d_i \cdot S\alpha_{i-1} \\ S\alpha_{i-1} \cdot S\theta_i & S\alpha_{i-1} \cdot C\theta_i & C\alpha_{i-1} & d_i \cdot C\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Practice



$\alpha_{i-1} \equiv angle(Z_{i-1}, Z_i)_{X_{i-1}}$

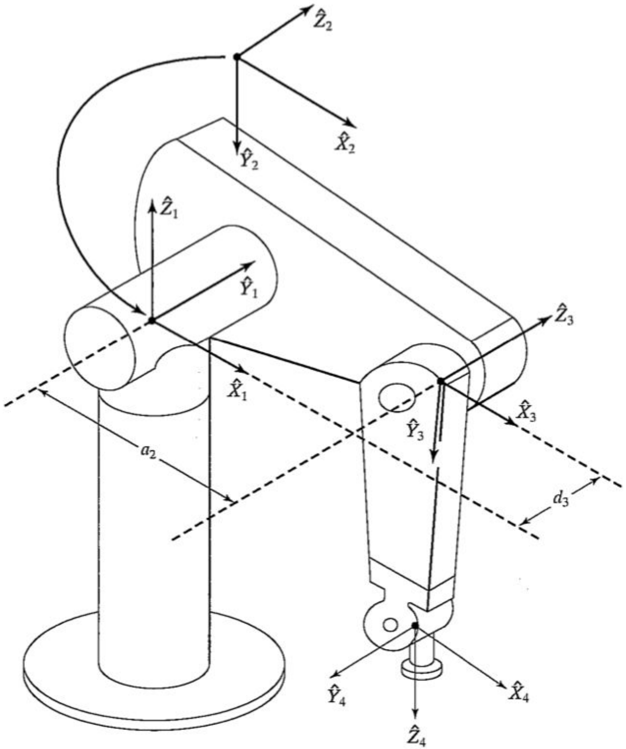
$d_i \equiv distance(X_{i-1}, X_i)_{Z_i}$

$a_{i-1} \equiv distance(Z_{i-1}, Z_i)_{X_{i-1}}$

$\theta_i \equiv angle(X_{i-1}, X_i)_{Z_i}$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2				
3				
4				
5				
6				

Practice



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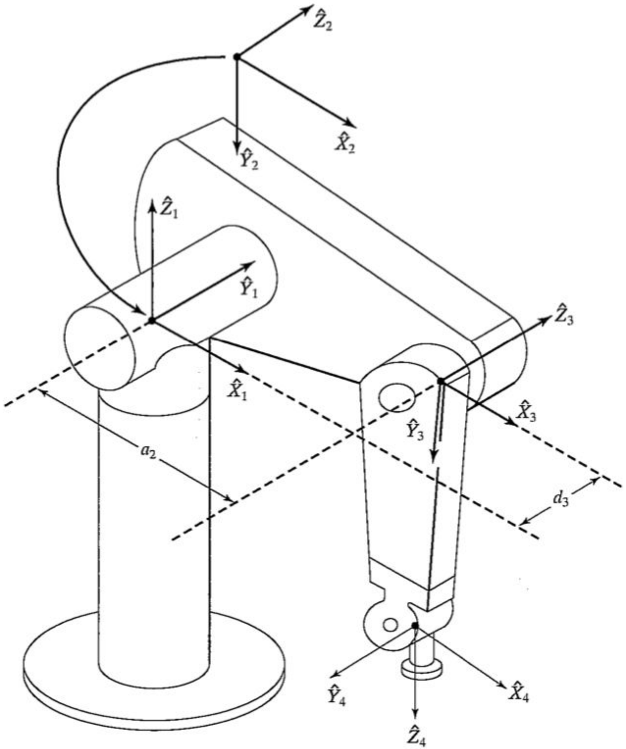
$a_{i-1} \equiv distance(Z_{i-1}, Z_i)_{X_{i-1}}$

$\theta_i \equiv angle(X_{i-1}, X_i)_{Z_i}$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-\pi/2$	0	0	θ_2
3	0	a_2	d_3	θ_3
4				
5				
6				

Practice



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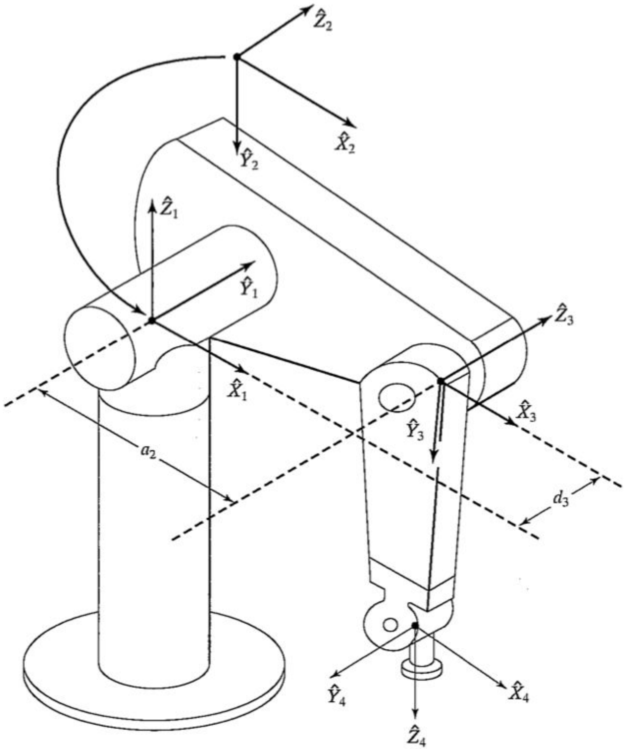
$a_{i-1} \equiv distance(Z_{i-1}, Z_i)_{X_{i-1}}$

$\theta_i \equiv angle(X_{i-1}, X_i)_{Z_i}$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-\pi/2$	0	0	θ_2
3	0	a_2	d_3	θ_3
4	$-\pi/2$	a_3	d_4	θ_4
5	$\pi/2$	0	0	θ_5
6				

Practice



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$d_i \equiv distance(X_{i-1}, X_i)_{Z_i}$

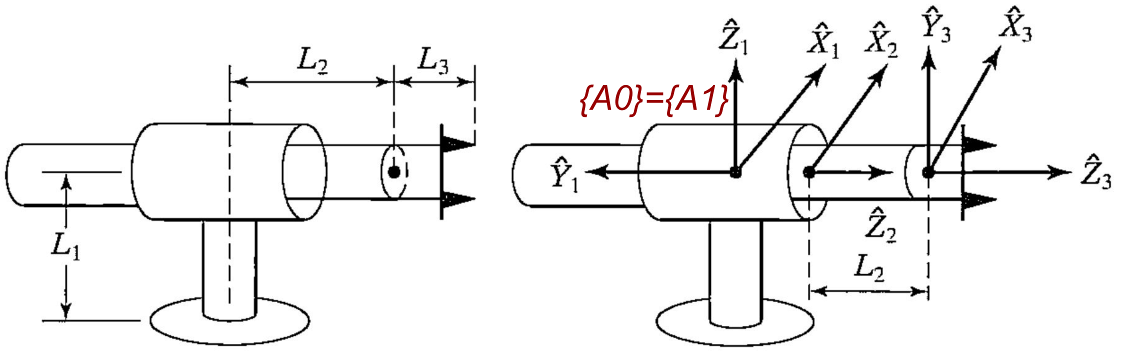
$a_{i-1} \equiv distance(Z_{i-1}, Z_i)_{X_{i-1}}$

$\theta_i \equiv angle(X_{i-1}, X_i)_{Z_i}$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-\pi/2$	0	0	θ_2
3	0	a_2	d_3	θ_3
4	$-\pi/2$	a_3	d_4	θ_4
5	$\pi/2$	0	0	θ_5
6	$-\pi/2$	0	0	θ_6

- Find ${}^{i-1}_{i}\mathbf{T}$ for $i = 1:6$
- What is ${}^{1}_{6}\mathbf{T}$?

Practice



$$\alpha_{i-1} \equiv angle(Z_{i-1}, Z_i)_{X_{i-1}}$$

$$a_{i-1} \equiv distance(Z_{i-1}, Z_i)_{X_{i-1}}$$

$$d_i \equiv distance(X_{i-1}, X_i)_{Z_i}$$

$$\theta_i \equiv angle(X_{i-1}, X_i)_{Z_i}$$

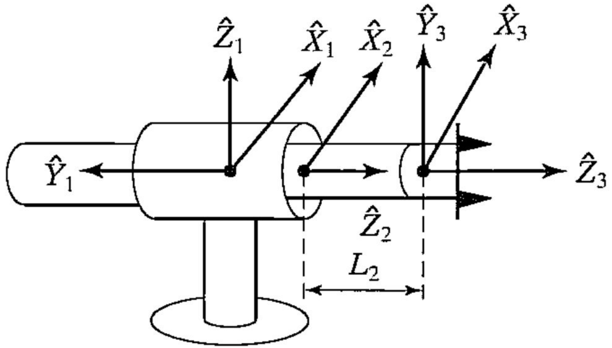
Say: $\Theta_1 = \pi/3$ and $\Theta_3 = \pi/6$

$d_2 = 0.5$ and $L_2 = 1$

- Find ${}^{i-1}_{i}\mathbf{T}$ for $i = 1:3$
- What is ${}^{1}_{3}\mathbf{T}$?
- What is ${}^{2}_{1}\mathbf{T}$?

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	Θ_1
2	$\pi/2$	0	d_2	0
3	0	0	L_2	Θ_3

Finding All Ts



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$\pi/2$	0	d_2	0
3	0	0	L_2	θ_3

Say: $\theta_1 = \pi/3$ and $\theta_3 = \pi/6$

$d_2 = 0.5$ and $L_2 = 1$

$$
{}^{i-1}_{i}\mathbf{T} = Rotate_{X_{i-1}}(\alpha_{i-1}) \times Translate_{X_{i-1}}(a_{i-1}) \times Rot_{Z_i}(\theta_i) \times Translate_{Z_i}(d_i)
$$

$$
= \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{i-1} \\ C\alpha_{i-1} \cdot S\theta_i & C\alpha_{i-1} \cdot C\theta_i & -S\alpha_{i-1} & -d_i \cdot S\alpha_{i-1} \\ S\alpha_{i-1} \cdot S\theta_i & S\alpha_{i-1} \cdot C\theta_i & C\alpha_{i-1} & d_i \cdot C\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{1}\mathbf{T} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(\theta_1 = \frac{\pi}{3})} \begin{bmatrix} 0.5 & -0.866 & 0 & 0 \\ 0.866 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(d_2 = 0.5)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}\mathbf{T} = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & 0 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(\theta_3 = \frac{\pi}{6}), L_2 = 1} \begin{bmatrix} 0.866 & -0.5 & 0 & 0 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Finding ${}^{1}_{3}T$ and ${}^{2}_{1}T$

R t

$$
{}^{1}_{2}\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(d_2=0.5)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}\mathbf{T} = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & 0 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(\theta_3=\frac{\pi}{6}), L_2=1} \begin{bmatrix} 0.866 & -0.5 & 0 & 0 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

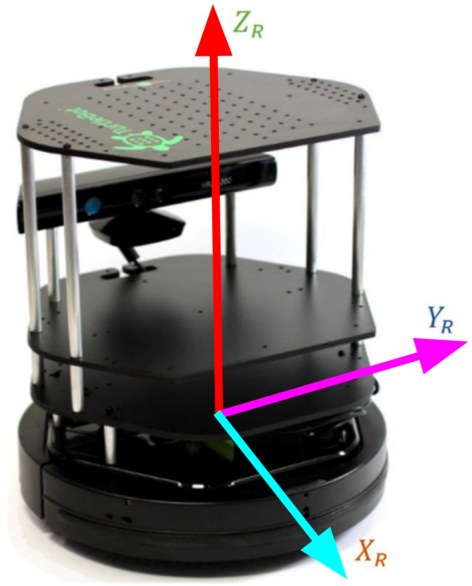
$$
{}^{1}_{3}\mathbf{T} = {}^{1}_{2}\mathbf{T} \cdot {}^{2}_{3}\mathbf{T} = \begin{bmatrix} 0.866 & -0.5 & 0 & 0 \\ 0 & 0 & -1 & -1.5 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

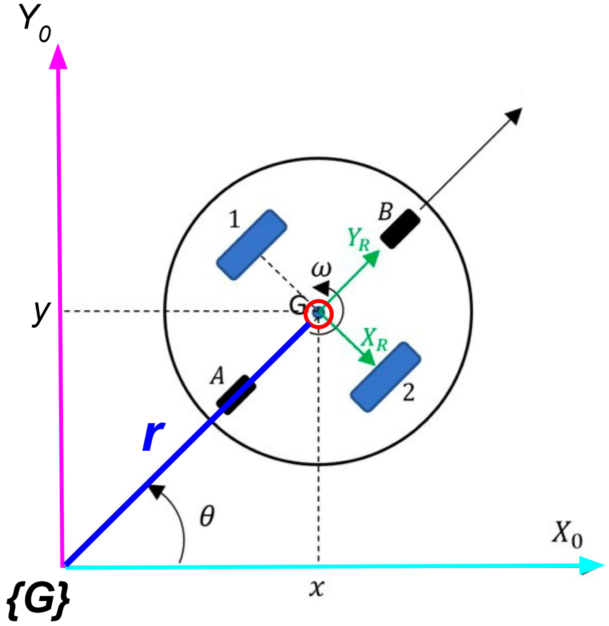
R' -R' t

$$
{}^{2}_{1}\mathbf{T} = {}^{1}_{2}\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

What is ${}^{3}_{0}\mathbf{T}$?

TurtleBot Kinematics





$$x = \mathbf{r} \cdot cos\theta$$

$$y = \mathbf{r} \cdot sin\theta$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} cos\theta & 0 \\ sin\theta & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}$$