Inverse Kinematics

EEL 4930/5934: Autonomous Robots

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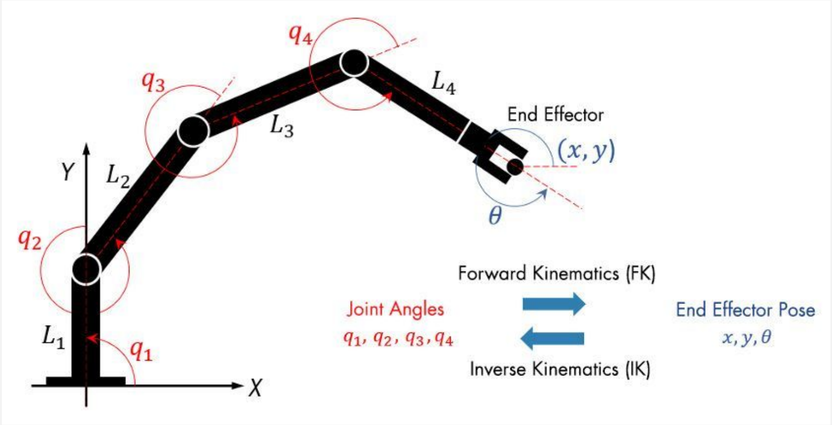
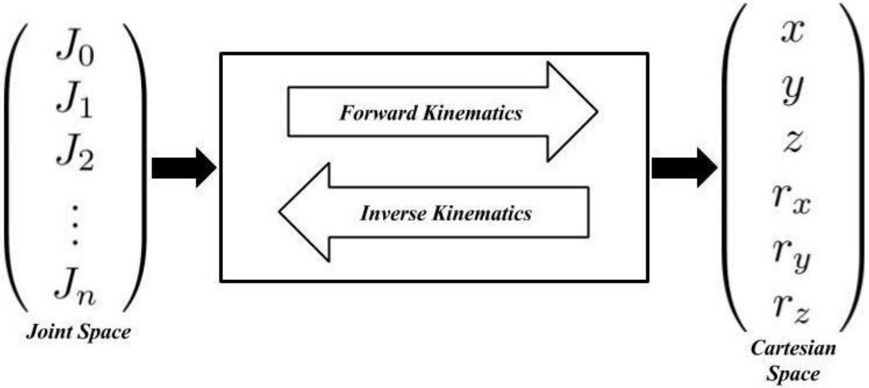
Md Jahidul Islam

Lecture 8

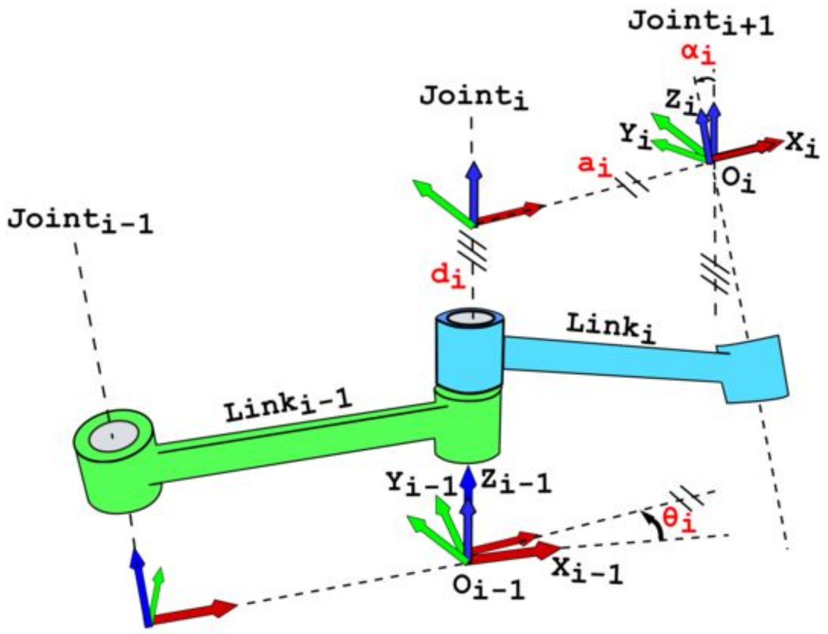
ECE | Florida
Electrical & Computer Engineering
UF | UNIVERSITY of FLORIDA

Forward Kinematics vs Inverse Kinematics





Manipulator Kinematics

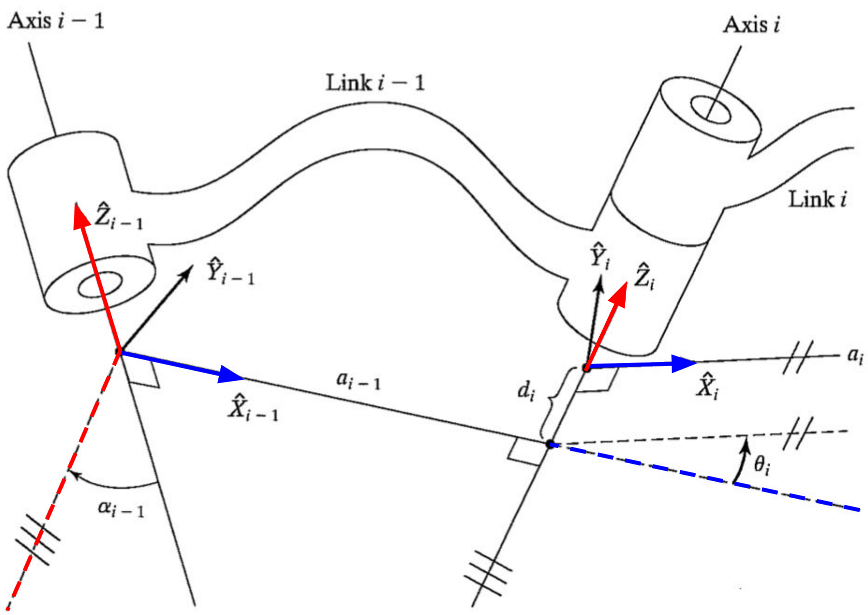


⇒ **Denavit-Hartenberg (DH) Notation**

- Standardizes kinematic notations
  - Coordinate frames for spatial linkages
  - Solving kinematic motion

- **Four parameters**
  - Link twist: $\boldsymbol{\alpha}$
  - Link length: ***a***
  - Link offset: ***d***
  - Joint angle: $\boldsymbol{\theta}$

DH Notation



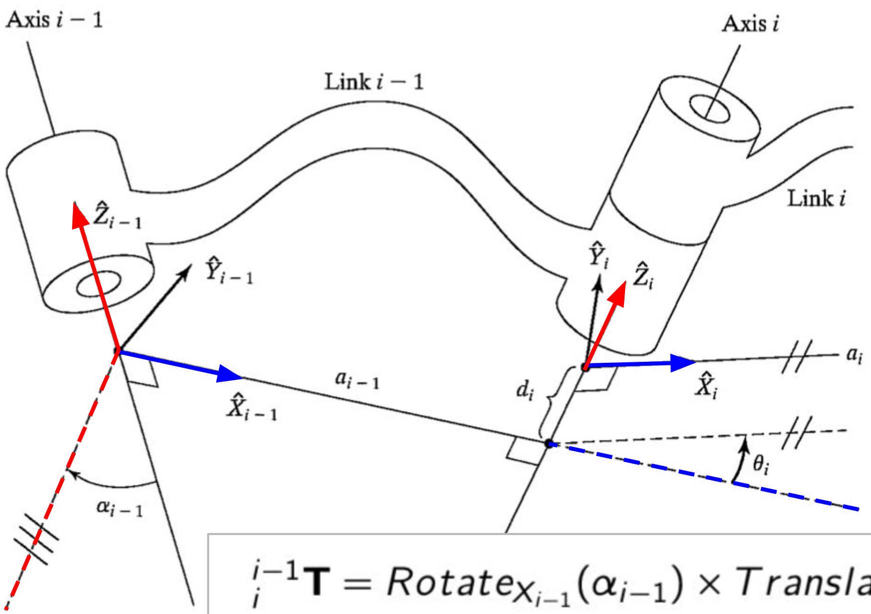
$$\alpha_{i-1} \equiv angle(Z_{i-1}, Z_i)_{X_{i-1}}$$

$$a_{i-1} \equiv distance(Z_{i-1}, Z_i)_{X_{i-1}}$$

$$d_i \equiv distance(X_{i-1}, X_i)_{Z_i}$$

$$\theta_i \equiv angle(X_{i-1}, X_i)_{Z_i}$$

Forward Kinematics: From {i} To {i-1}

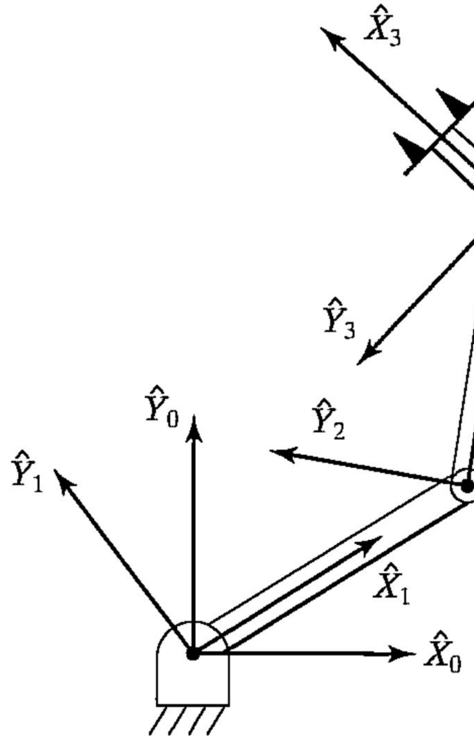


1. Translate $\boldsymbol{d_i}$ *along* $\boldsymbol{Z_i}$
2. Rotate $\boldsymbol{\theta_i}$ *around* $\boldsymbol{Z_i}$
3. Translate $\boldsymbol{a_{i-1}}$ *along* $\boldsymbol{X_{i-1}}$
4. Rotate $\boldsymbol{\alpha_{i-1}}$ *around* $\boldsymbol{X_{i-1}}$

$$
{}^{i-1}_{i}\mathbf{T} = Rotate_{X_{i-1}}(\alpha_{i-1}) \times Translate_{X_{i-1}}(a_{i-1}) \times Rot_{Z_i}(\theta_i) \times Translate_{Z_i}(d_i)
$$

$$
= \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{i-1} \\ C\alpha_{i-1} \cdot S\theta_i & C\alpha_{i-1} \cdot C\theta_i & -S\alpha_{i-1} & -d_i \cdot S\alpha_{i-1} \\ S\alpha_{i-1} \cdot S\theta_i & S\alpha_{i-1} \cdot C\theta_i & C\alpha_{i-1} & d_i \cdot C\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

DH Notation: Example 1



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	**0**	**L_1**	**0**	**θ_2**
3	**0**	**L_2**	**0**	**θ_3**

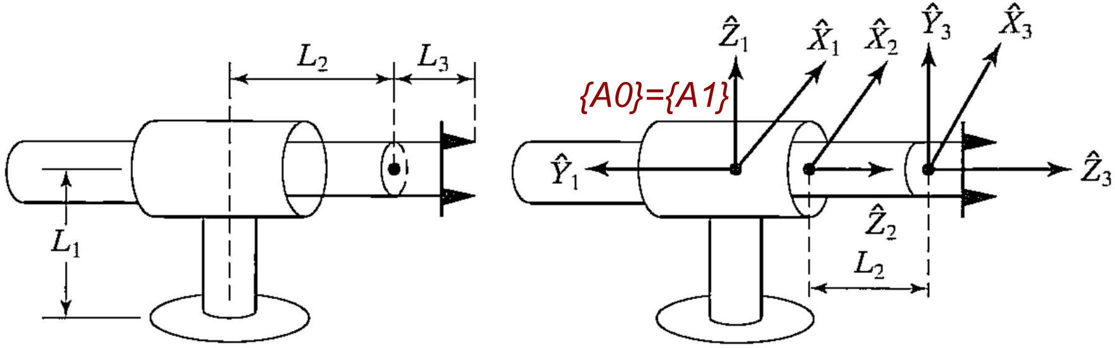
$$\alpha_{i-1} \equiv angle(Z_{i-1}, Z_i)_{X_{i-1}}$$

$$a_{i-1} \equiv distance(Z_{i-1}, Z_i)_{X_{i-1}}$$

$$d_i \equiv distance(X_{i-1}, X_i)_{Z_i}$$

$$\theta_i \equiv angle(X_{i-1}, X_i)_{Z_i}$$

DH Notation: Example 2



$\alpha_{i-1} \equiv angle(Z_{i-1}, Z_i)_{X_{i-1}}$

$a_{i-1} \equiv distance(Z_{i-1}, Z_i)_{X_{i-1}}$

$d_i \equiv distance(X_{i-1}, X_i)_{Z_i}$

$\theta_i \equiv angle(X_{i-1}, X_i)_{Z_i}$

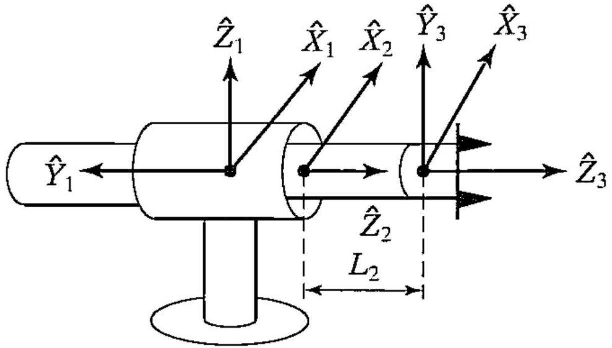
Say: $\Theta_1 = \pi/3$ and $\Theta_3 = \pi/6$

$d_2 = 0.5$ and $L_2 = 1$

- Find ${}^{i-1}_{i}\mathbf{T}$ for $i = 1:3$
- What is ${}^{1}_{3}\mathbf{T}$?
- What is ${}^{2}_{1}\mathbf{T}$?

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	Θ_1
2	$\pi/2$	0	d_2	0
3	0	0	L_2	Θ_3

Finding All Ts



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	Θ_1
2	$\pi/2$	0	d_2	0
3	0	0	L_2	Θ_3

Say: $\Theta_1 = \pi/3$ and $\Theta_3 = \pi/6$

$d_2 = 0.5$ and $L_2 = 1$

$$ {}^{i-1}_{i}\mathbf{T} = Rotate_{X_{i-1}}(\alpha_{i-1}) \times Translate_{X_{i-1}}(a_{i-1}) \times Rot_{Z_i}(\theta_i) \times Translate_{Z_i}(d_i) $$
$$ = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{i-1} \\ C\alpha_{i-1}\cdot S\theta_i & C\alpha_{i-1}\cdot C\theta_i & -S\alpha_{i-1} & -d_i\cdot S\alpha_{i-1} \\ S\alpha_{i-1}\cdot S\theta_i & S\alpha_{i-1}\cdot C\theta_i & C\alpha_{i-1} & d_i\cdot C\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{1}\mathbf{T} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(\theta_1=\frac{\pi}{3})} \begin{bmatrix} 0.5 & -0.866 & 0 & 0 \\ 0.866 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(d_2=0.5)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}\mathbf{T} = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & 0 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(\theta_3=\frac{\pi}{6}),L_2=1} \begin{bmatrix} 0.866 & -0.5 & 0 & 0 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Finding ${}^{1}_{3}T$ and ${}^{2}_{1}T$

R, t

$${}^{1}_{2}\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(d_2=0.5)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

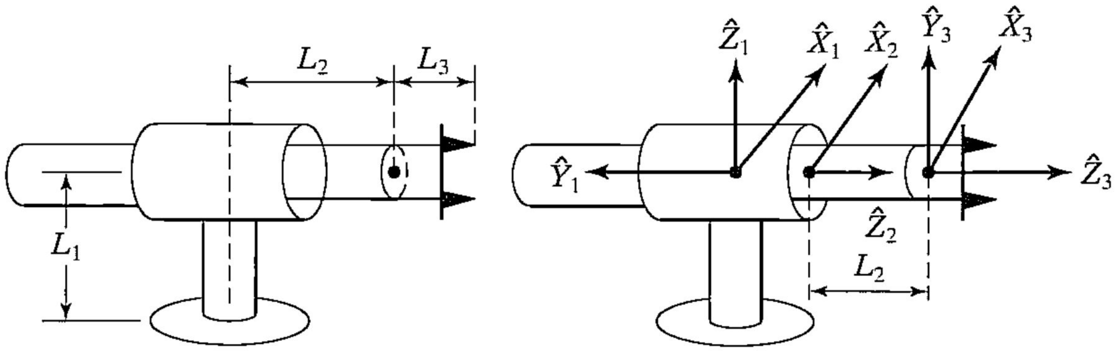
$${}^{2}_{3}\mathbf{T} = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & 0 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(\theta_3=\frac{\pi}{6}), L_2=1} \begin{bmatrix} 0.866 & -0.5 & 0 & 0 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

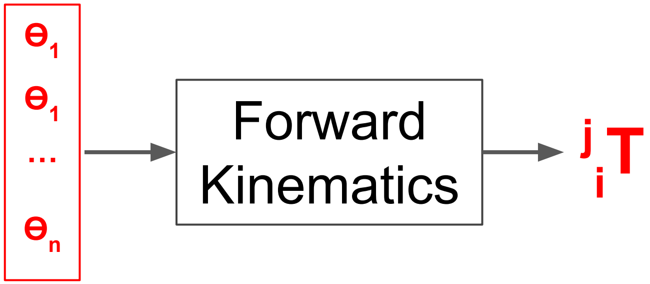
R', -R' t

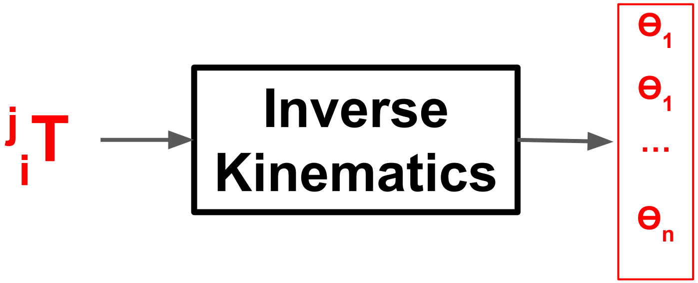
$${}^{2}_{1}\mathbf{T} = {}^{1}_{2}\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}_{3}\mathbf{T} = {}^{1}_{2}\mathbf{T} \cdot {}^{2}_{3}\mathbf{T} = \begin{bmatrix} 0.866 & -0.5 & 0 & 0 \\ 0 & 0 & -1 & -1.5 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics







Solvability & Workspace

$^{j}_{i}T$ → **Inverse Kinematics** → θ_1, θ_1, …, θ_n

⇒ Is there a solution?

- If yes, how many solutions?
- If no, why not?

⇒ How to find them?

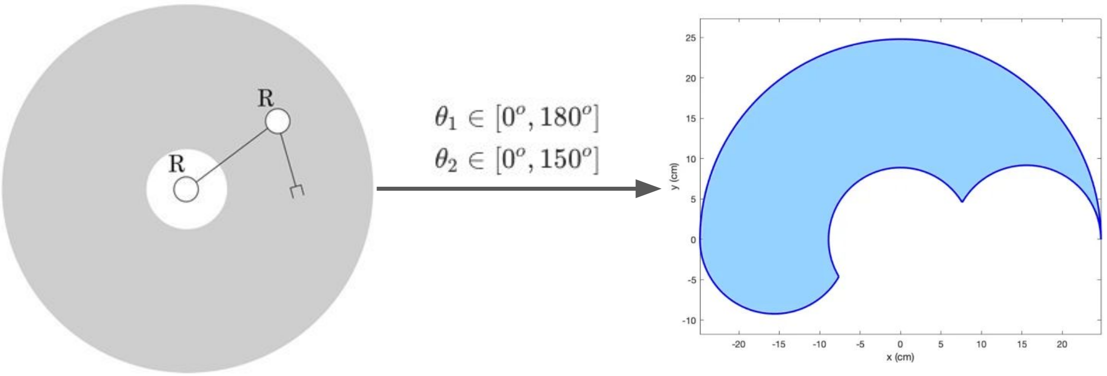
- Numerically or Analytically
- Some may not be feasible!

⇒ Workspace of a manipulator or end-effector

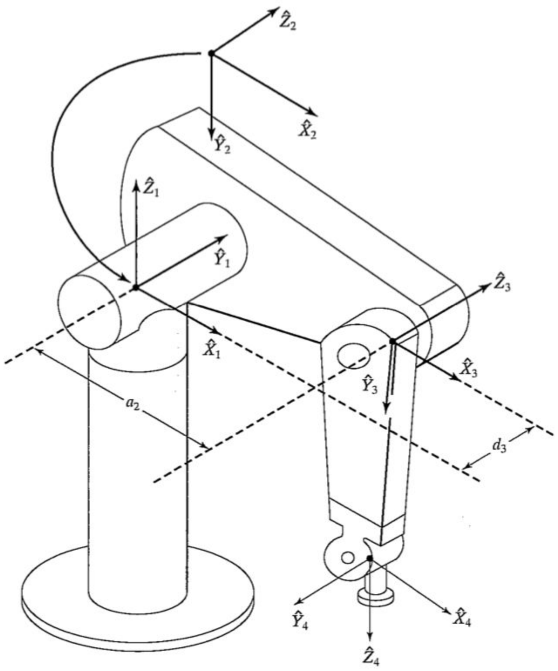
- Volume of space that the end-effector can reach

⇒ Dexterous workspace

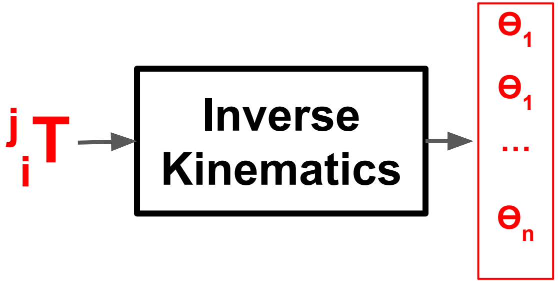
- Volume of space that the end-effector can reach with all orientations



Kinematic Solutions: PUMA 560

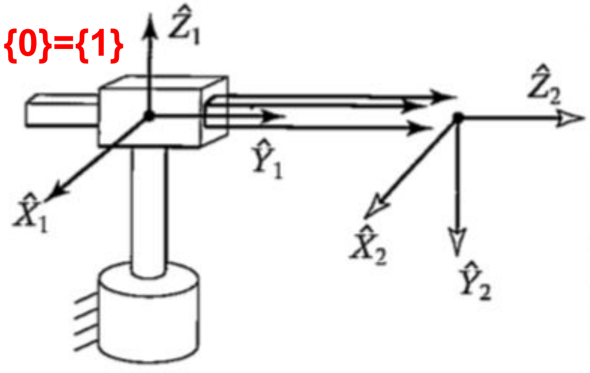


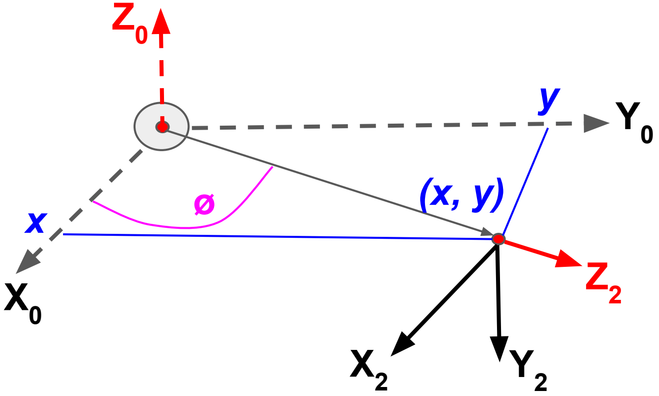
i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-\pi/2$	0	0	θ_2
3	0	a_2	d_3	θ_3
4	$-\pi/2$	a_3	d_4	θ_4
5	$\pi/2$	0	0	θ_5
6	$-\pi/2$	0	0	θ_6



a_i	# of solutions
$a_1 = a_3 = a_5 = 0$	≤ 4
$a_3 = a_5 = 0$	≤ 8
$a_3 = 0$	≤ 16
All $a_3 \neq 0$	≤ 16

Practice #1: Geometric Solution





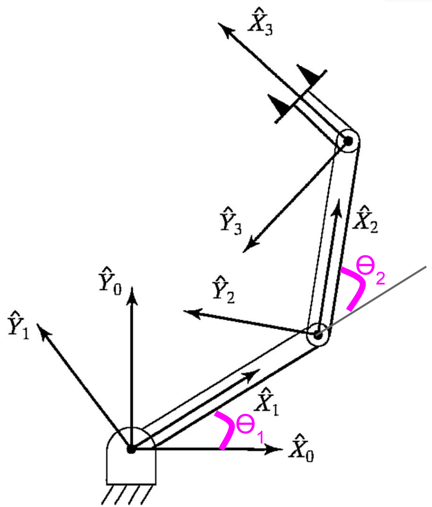
Given $${}^0_2\mathbf{T} = \left[\begin{array}{ccc|c} & {}^0_2R & & {}^0P_{2org} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} \sin\phi & 0 & \cos\phi & x \\ -\cos\phi & 0 & \sin\phi & y \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find ø

$${}^0P_{2org} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad {}^0_2R = \begin{bmatrix} \frac{y}{x^2+y^2} & 0 & \frac{x}{x^2+y^2} \\ -\frac{x}{x^2+y^2} & 0 & \frac{y}{x^2+y^2} \\ 0 & -1 & 0 \end{bmatrix}$$

$$\sin\phi = \frac{y}{x^2+y^2}$$
$$\cos\phi = \frac{x}{x^2+y^2}$$

Practice #2: Algebraic Solution



$${}^{i-1}_{i}\mathbf{T} = Rotate_{X_{i-1}}(\alpha_{i-1}) \times Translate_{X_{i-1}}(a_{i-1}) \times Rot_{Z_i}(\theta_i) \times Translate_{Z_i}(d_i)$$

$$= \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{i-1} \\ C\alpha_{i-1}\cdot S\theta_i & C\alpha_{i-1}\cdot C\theta_i & -S\alpha_{i-1} & -d_i\cdot S\alpha_{i-1} \\ S\alpha_{i-1}\cdot S\theta_i & S\alpha_{i-1}\cdot C\theta_i & C\alpha_{i-1} & d_i\cdot C\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}_{3}\mathbf{T} = {}^{0}_{1}\mathbf{T}\cdot{}^{1}_{2}\mathbf{T}\cdot{}^{2}_{3}\mathbf{T}$$

$$= \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & L_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & L_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

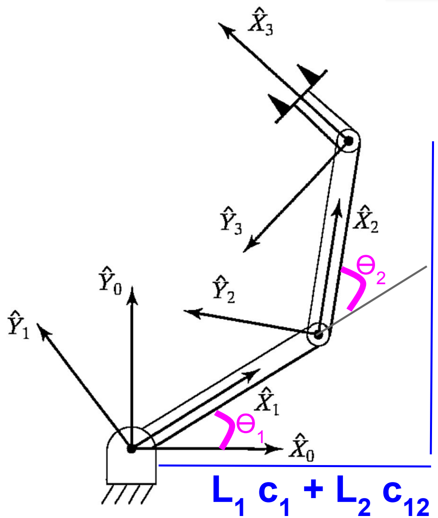
$$= \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c(\theta_2+\theta_3) & -s(\theta_2+\theta_3) & 0 & L_2c\theta_2 + L_1 \\ s(\theta_2+\theta_3) & c(\theta_2+\theta_3) & 0 & L_2s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_{23} & -s_{23} & 0 & L_2c_2 + L_1 \\ s_{23} & c_{23} & 0 & L_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{123} & -s_{123} & 0 & L_1c_1 + L_2c_{12} \\ s_{123} & c_{123} & 0 & L_1s_1 + L_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

Practice #2: Finding Θ_2



$$ {}^0_3\mathbf{T} = {}^0_1\mathbf{T} \cdot {}^1_2\mathbf{T} \cdot {}^2_3\mathbf{T} $$

$$ = \begin{bmatrix} c_{123} & -s_{123} & 0 & L_1c_1 + L_2c_{12} \\ s_{123} & c_{123} & 0 & L_1s_1 + L_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} c\phi & -s\phi & 0 & x \\ s\phi & c\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ x^2 + y^2 = L_1^2 + L_2^2 + 2L_1L_2c_2 $$

$$ c2 = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2} $$

$$ s2 = \pm\sqrt{1 - c_2^2} $$

$$ \theta_2 = Atan2(s2,\ c2) $$

$$ x = L_1c_1 + L_2c_{12} = L_1c_1 + L_2c_1c_2 - L_2s_1s_2 = (L_1 + L_2c_2)c_1 - (L_2s_2)s_1 $$

$$ y = L_1s_1 + L_2s_{12} = L_1s_1 + L_2s_1c_2 + L_2c_1s_2 = (L_1 + L_2c_2)s_1 + (L_2s_2)c_1 $$

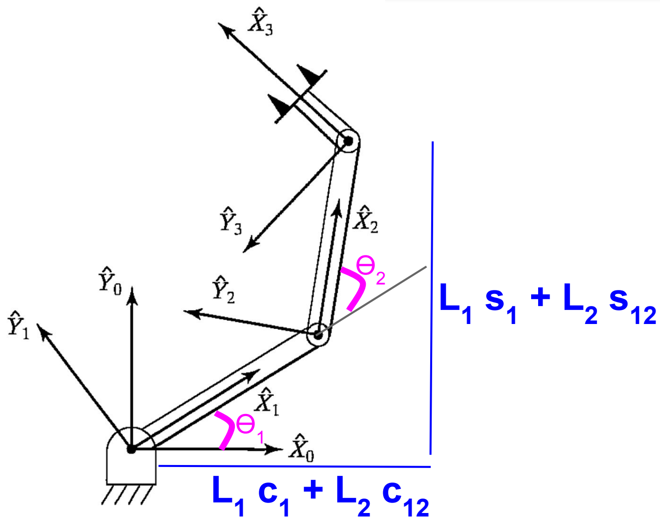
$$ x = k_1c_1 - k_2s_1 $$
$$ y = k_1s_1 + k_2c_1 $$

$$ k_1 = r\cos\gamma $$
$$ k_2 = r\sin\gamma $$
$$ r = \sqrt{k_1^2 + k_2^2} $$

$$ \frac{x}{r} = c\gamma c_1 - s\gamma s_1 = \cos(\gamma + \theta_1) $$
$$ \frac{y}{r} = c\gamma s_1 + s\gamma c_1 = \sin(\gamma + \theta_1) $$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	Θ_1
2	0	L_1	0	Θ_2
3	0	L_2	0	Θ_3

Practice #2: Finding All ϴs



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

$$x = L_1c_1 + L_2c_{12} = L_1c_1 + L_2c_1c_2 - L_2s_1s_2 = (L_1 + L_2c_2)c_1 - (L_2s_2)s_1$$
$$y = L_1s_1 + L_2s_{12} = L_1s_1 + L_2s_1c_2 + L_2c_1s_2 = (L_1 + L_2c_2)s_1 + (L_2s_2)c_1$$

$$x = k_1c_1 - k_2s_1$$
$$y = k_1s_1 + k_2c_1$$

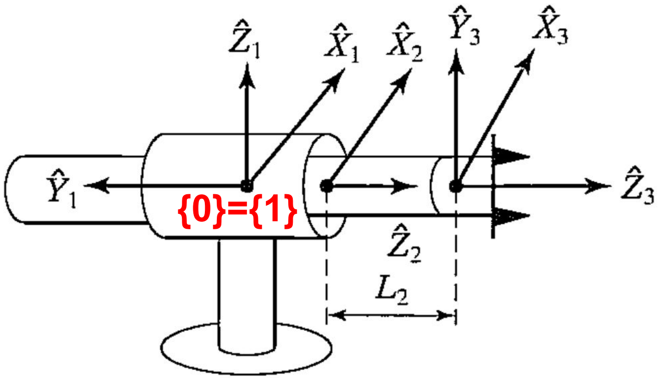
$$k_1 = r\cos\gamma$$
$$k_2 = r\sin\gamma$$
$$r = \sqrt{k_1^2 + k_2^2}$$

$$\frac{x}{r} = c\gamma c_1 - s\gamma s_1 = \cos(\gamma + \theta_1)$$
$$\frac{y}{r} = c\gamma s_1 + s\gamma c_1 = \sin(\gamma + \theta_1)$$

$$\gamma + \theta_1 = Atan2(\frac{y}{r}, \frac{x}{r}) = Atan2(y, x)$$
$$\theta_1 = Atan2(y, x) - Atan2(k_2, k_1)$$

$$\phi = \theta_1 + \theta_2 + \theta_3 = Atan2(s_{123}, c_{123})$$
$$\theta_3 = Atan2(s_{123}, c_{123}) - \theta_1 - \theta_2$$

Practice #3: In-class Exercise



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$\pi/2$	0	d_2	0
3	0	0	$L_2 = 1$	θ_3

$${}^{i-1}_{i}\mathbf{T} = Rotate_{X_{i-1}}(\alpha_{i-1}) \times Translate_{X_{i-1}}(a_{i-1}) \times Rot_{Z_i}(\theta_i) \times Translate_{Z_i}(d_i)$$

$$= \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{i-1} \\ C\alpha_{i-1} \cdot S\theta_i & C\alpha_{i-1} \cdot C\theta_i & -S\alpha_{i-1} & -d_i \cdot S\alpha_{i-1} \\ S\alpha_{i-1} \cdot S\theta_i & S\alpha_{i-1} \cdot C\theta_i & C\alpha_{i-1} & d_i \cdot C\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

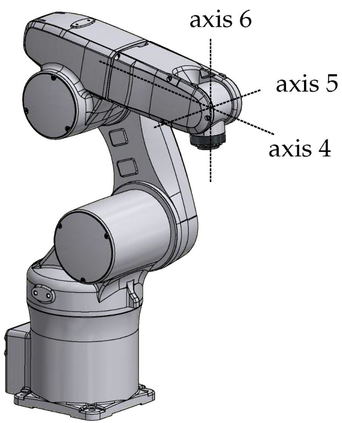
$${}^{0}_{3}\mathbf{T} = {}^{0}_{1}\mathbf{T} \cdot {}^{1}_{2}\mathbf{T} \cdot {}^{2}_{3}\mathbf{T}$$

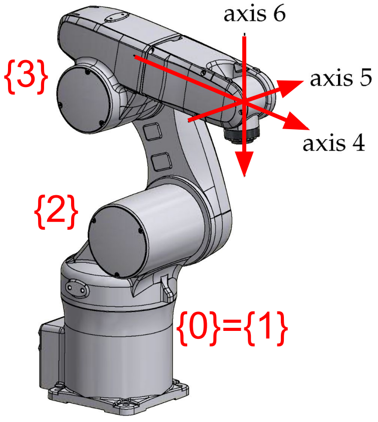
$$= \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & 0 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & L_2 = 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.433 & -0.25 & 0.866 & 1.30 \\ 0.75 & -0.433 & -0.5 & -0.75 \\ 0.5 & 0.866 & 0. & 0. \\ 0. & 0. & 0. & 1. \end{bmatrix} \quad \textbf{(given)}$$

Find θ_1, d_2, θ_3

Practice #4: Pieper Robot





6R manipulator: Axis 4, 5, 6 intersect

$${}^{0}_{6}\mathbf{P} = {}^{0}_{5}\mathbf{P} = {}^{0}_{4}\mathbf{P}$$

$${}^{0}_{4}\mathbf{T} = \left[\begin{array}{ccc|c} & {}^{0}_{4}\mathbf{R} & & {}^{0}\mathbf{P}_{4org} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Find θ_1, θ_2, θ_3

$${}^{i-1}_{i}\mathbf{T} = Rotate_{X_{i-1}}(\alpha_{i-1}) \times Translate_{X_{i-1}}(a_{i-1}) \times Rot_{Z_i}(\theta_i) \times Translate_{Z_i}(d_i)$$

$$= \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{i-1} \\ C\alpha_{i-1}\cdot S\theta_i & C\alpha_{i-1}\cdot C\theta_i & -S\alpha_{i-1} & -d_i\cdot S\alpha_{i-1} \\ S\alpha_{i-1}\cdot S\theta_i & S\alpha_{i-1}\cdot C\theta_i & C\alpha_{i-1} & d_i\cdot C\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

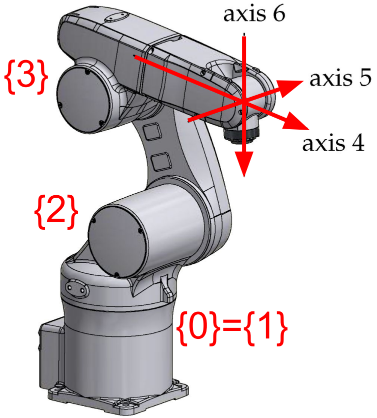
$${}^{0}\mathbf{P}_{4org} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = {}^{0}_{1}\mathbf{T}\cdot{}^{1}_{2}\mathbf{T}\cdot{}^{2}_{3}\mathbf{T}\cdot{}^{3}\mathbf{P}_{4org} = {}^{0}_{1}\mathbf{T}\cdot{}^{1}_{2}\mathbf{T}\cdot{}^{2}_{3}\mathbf{T}\cdot\begin{bmatrix} a_3 \\ -d_4 s\alpha_{i-1} \\ d_4 c\alpha_{i-1} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = {}^{0}_{1}\mathbf{T}\cdot{}^{1}_{2}\mathbf{T}\cdot{}^{2}_{3}\mathbf{T}\cdot{}^{3}\mathbf{P}_{4org} = {}^{0}_{1}\mathbf{T}\cdot{}^{1}_{2}\mathbf{T}\cdot\begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}$$

$$= {}^{0}_{1}\mathbf{T}\cdot\begin{bmatrix} g_1(\theta_2,\theta_3) \\ g_2(\theta_2,\theta_3) \\ g_3(\theta_2,\theta_3) \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 g_1 - c_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix}$$

Practice #4: Process of Elimination



6R manipulator: Axis 4, 5, 6 intersect

$${}^0_6\mathbf{P} = {}^0_5\mathbf{P} = {}^0_4\mathbf{P}$$

$${}^0_4\mathbf{T} = \left[\begin{array}{ccc|c} & {}^0_4\mathbf{R} & & {}^0\mathbf{P}_{4org} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Find θ_1, θ_2, θ_3

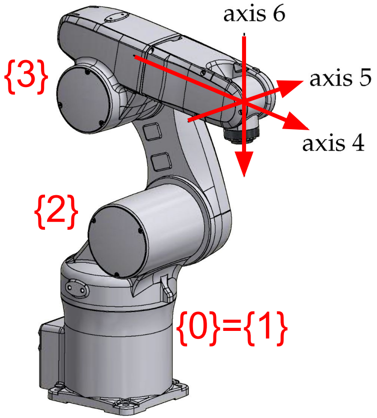
$${}^0\mathbf{P}_{4org} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = {}^0_1\mathbf{T} \cdot {}^1_2\mathbf{T} \cdot {}^2_3\mathbf{T} \cdot {}^3\mathbf{P}_{4org} = {}^0_1\mathbf{T} \cdot {}^1_2\mathbf{T} \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 1 \end{bmatrix} = {}^0_1\mathbf{T} \cdot \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 g_1 - c_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix}$$

$$\begin{aligned} f_1 &= a_3 c_3 + d_4 s\alpha_3 s_3 + a_2 \\ f_2 &= a_3 c\alpha_2 s\alpha_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2 \\ f_3 &= a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2 \end{aligned}$$

$$\begin{aligned} g_1 &= c_2 f_1 - s_2 f_2 + a_1 \\ g_2 &= s_2 c\alpha_1 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1 \\ g_3 &= s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1 \end{aligned}$$

$$\begin{aligned} r = x^2 + y^2 + z^2 &= g_1^2 + g_2^2 + g_3^2 \\ &= f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2 d_2 f_3 + 2 a_1 (c_2 f_1 - s_2 f_2) \end{aligned}$$

Practice #4: Finding $\theta_3 \Rightarrow \theta_2 \Rightarrow \theta_1$



6R manipulator: Axis 4, 5, 6 intersect

$$ {}^0_6\mathbf{P} = {}^0_5\mathbf{P} = {}^0_4\mathbf{P} $$

$$ {}^0_4\mathbf{T} = \begin{bmatrix} {}^0_4\mathbf{R} & {}^0\mathbf{P}_{4org} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

Find $\theta_1, \theta_2, \theta_3$

$$ {}^0\mathbf{P}_{4org} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = {}^0_1\mathbf{T} \cdot {}^1_2\mathbf{T} \cdot {}^2_3\mathbf{T} \cdot {}^3\mathbf{P}_{4org} = {}^0_1\mathbf{T} \cdot {}^1_2\mathbf{T} \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 1 \end{bmatrix} = {}^0_1\mathbf{T} \cdot \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 g_1 - c_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix} $$

Eliminate variable θ_1 and formulate two equations to solve θ_2 and θ_3

$$ r = 2a_1(k_1 c_2 + k_2 s_2) + k_3 $$
$$ z = g_3 = s\alpha_1(k_1 s_2 - k_2 c_2) + k_4 $$

$$ k_1 = f_1 $$
$$ k_2 = -f_2 $$
$$ k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 $$
$$ k_4 = f_3 c\alpha_1 + d_2 c\alpha_1 $$

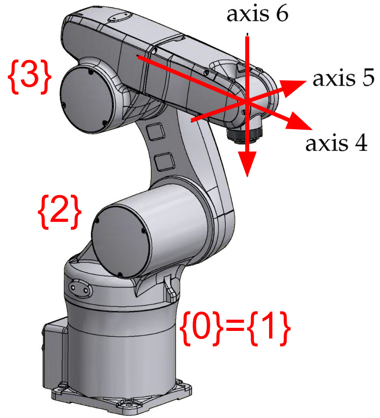
⇒ Special cases (solve for θ_3):

- If $a_1 = 0$, $r = k_3$. **2 values**
- If $s\alpha_1 = 0$, $z = k_4$. **2 values**
- Otherwise $\frac{(r-k_3)^2}{4a_1^2} + \frac{(z-k_4)^2}{s^2\alpha_1} = k_1^2 + k_2^2$. **4 values.**

⇒ With θ_3 solve for θ_2

⇒ Then solve for θ_1

Practice #4: Finding θ_4, θ_5, and θ_6



6R manipulator: Axis 4, 5, 6 intersect

$$^0_6P = {^0_5P} = {^0_4P}$$

$$^0_4T = \begin{bmatrix} ^0_4R & ^0P_{4org} \\ 0 \; 0 \; 0 & 1 \end{bmatrix}$$

We have θ_1, θ_2, θ_3 find θ_4, θ_5, θ_6

$$^0_6R = {^0_1R} \cdot {^1_2R} \cdot {^2_3R} \cdot {^3_4R} \cdot {^4_5R} \cdot {^5_6R}$$

$$^2_3R^T \cdot {^1_2R^T} \cdot {^0_1R^T} \cdot {^0_6R} = {^3_4R} \cdot {^4_5R} \cdot {^5_6R}$$

$$R = {^3_4R} \cdot {^4_5R} \cdot {^5_6R}$$

9 equations: 3 unknowns!

See Craig's book example 4.6

Further Reading

⇒ Craig's book

- Example 4.7: PUMA-560 robot inverse kinematics
- Exercises: 4.2, 4.5, 4.10, 4.11, 4.16

⇒ A few good resources:

- https://motion.cs.illinois.edu/RoboticSystems/InverseKinematics.html
- http://docs.ros.org/en/kinetic/api/moveit_tutorials/html/doc/robot_model_and_robot_state/robot_model_and_robot_state_tutorial.html
- http://docs.ros.org/en/indigo/api/moveit_tutorials/html/doc/pr2_tutorials/kinematics/src/doc/kinematic_model_tutorial.html
- https://www.rosroboticslearning.com/forward-kinematics
- https://www.rosroboticslearning.com/inverse-kinematics